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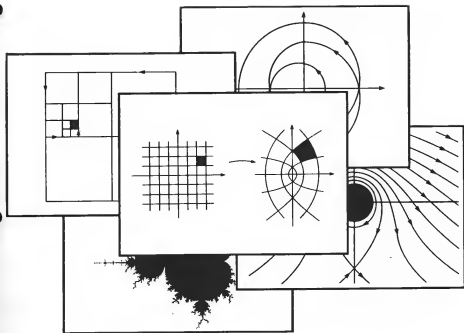
COMPLEX ANALYSIS

HANDBOOK ONE

COMPLEX ANALYSIS   COMPLEX ANALYSIS   COMPL

CONTENTS

PART I: REFERENCE MATERIAL

Greek alphabet

α	A	alpha	ι	I	iota	ρ	P	rho
β	B	beta	κ	K	kappa	σ	Σ	sigma
γ	Γ	gamma	λ	Λ	lambda	τ	T	tau
δ	Δ	delta	μ	M	mu	υ	Υ	upsilon
ε	E	epsilon	ν	N	nu	ϕ	Φ	phi
ζ	Z	zeta	ξ	Ξ	xi	χ	X	chi
η	H	eta	o	O	omicron	ψ	Ψ	psi
θ	Θ	theta	π	Π	pi	ω	Ω	omega

Mathematical language

1. In mathematics we commonly use implications such as 'if P then Q', where P and Q are statements which can be either true or false. The statement P is the hypothesis and the statement Q is the conclusion. An implication is *true* if the conclusion is true whenever the hypothesis is true and false otherwise. For example, the following implication is true:

   if x is positive, then $x + 1$ is positive.

There are various equivalent ways of stating such an implication:

(a) if $x > 0$, then $x + 1 > 0$;

(b) $x > 0 \implies x + 1 > 0$;

(c) for all $x > 0$, we have $x + 1 > 0$;

(d) $x + 1 > 0$, for all $x > 0$;

(e) $x + 1 > 0$, whenever $x > 0$;

(f) for $x + 1$ to be positive, it is sufficient that x be positive.

2. The *converse* of an implication is obtained by exchanging the hypothesis and the conclusion. For example, the converse of the (true) implication

   if $x > 0$, then $x + 1 > 0$

is

   if $x + 1 > 0$, then $x > 0$,

which is false (try $x = 0$).

3. An equivalence consists of an implication 'if P then Q' *and its
converse* 'if Q then P'. The equivalence is true if both these implications are true.
For example, the following equivalence is true:

$x > 0$ is equivalent to $2x > 0$.

It could alternatively be stated as follows:

$x > 0 \iff 2x > 0$;

$x > 0$ if and only if $2x > 0$;

$x > 0$ is necessary and sufficient for $2x > 0$.

4. There are three ways of proving implications:

   (a) **direct proof**: we begin by assuming that the hypothesis is true and then argue directly to show that the conclusion is true;

   (b) **proof by contraposition**: we begin by assuming that the conclusion is false and then argue directly to show that the hypothesis is false;

   (c) **proof by contradiction**: we begin by assuming that the conclusion is false, and then argue from both to obtain a contradiction.

Where possible, types (a) or (b) are preferable, since they establish a direct link between hypothesis and conclusion, but type (c) is often convenient and sometimes essential.

Set notation

Notation	Meaning
$\{x, y, \ldots, z\}$	The set of elements listed in $\{\ldots\}$
$\{x : \ldots\}$	The set of all x such that $\ldots$ holds
$x \in A$	x belongs to A
$x \notin A$	x does not belong to A
$A \subseteq B$	A is a subset of B: each element of A belongs to B
$A = B$	A is equal to B: $A \subseteq B$ and $B \subseteq A$
$A \subset B$	A is a proper subset of B: $A \subseteq B$ but $A \neq B$
$A \cup B$	A union B: the set of all elements which belong to A or B (or both)
$A \cap B$	A intersection B: the set of all elements which belong to both A and B
$A - B$	A minus B: the set of all elements of A which do not belong to B
$\varnothing$	The empty set

Some texts use
$A \subset B$
to mean
$A \subseteq B$.

Real numbers

1. Definitions

A **real number** is a number which can be represented by a decimal of the form

$$\pm a_0.a_1 a_2 a_3 \ldots,$$

where a_0 is a non-negative integer and a_1, a_2, a_3, ... are digits. **Rational numbers** (ratios of integers) are represented by recurring decimals and **irrational numbers** are represented by non-recurring decimals. Real numbers are often represented by points on a line, called the **real line**.

2. Some important subsets of the real numbers

Symbol	Subset
$\mathbb{N}$	The set of all natural numbers: $1, 2, 3, \ldots$
$\mathbb{Z}$	The set of all integers: $0, \pm 1, \pm 2, \pm 3, \ldots$
$\mathbb{Q}$	The set of all rational numbers (numbers of the form p/q, where $p, q \in \mathbb{Z}, q \neq 0$)
$\mathbb{R}$	The set of all real numbers
$]a, b[$	$\{x : a < x < b\}$, the open interval from a to b
$[a, b]$	$\{x : a \leq x \leq b\}$, the closed interval from a to b
$]a, b]$	$\{x : a < x \leq b\}$
$[a, \infty[$	$\{x : x \geq a\}$
$]a, \infty[$	$\{x : x > a\}$
$]-\infty, a[$	$\{x : x < a\}$

3. Upper and lower bounds

Suppose that A is a non-empty subset of $\mathbb{R}$. Then A is bounded above if there is a real number M such that

$$x \leq M, \quad \text{for all } x \in A.$$

The number M is called an upper bound of A. Clearly, any number bigger than M is also an upper bound of A. A lower bound of A is defined similarly.

Among all upper bounds of A, the smallest (which always exists if A is bounded above) is called the least upper bound of A, or the supremum of A, written sup A. The greatest lower bound, or the infimum, of A, written inf A, is defined similarly.

If A has infinitely many elements, then sup A and inf A may or may not belong to A. In contrast, if A has finitely many elements, then sup A and inf A are the largest and the smallest elements of A, respectively, and therefore are members of A.

If sup A belongs to A, then we may use the alternative notation max A for sup A. Similarly, if inf A belongs to A, then we may denote it by min A.

Example If $A = [-1, 1[$, then

$$\sup A = 1 \quad \text{and} \quad \inf A = \min A = -1.$$

If $B = \{-1, 0, 1, 2\}$, then

$$\sup B = \max B = 2 \quad \text{and} \quad \inf B = \min B = -1.$$

4. Inequalities

Rules for rearranging inequalities

(a) $a < b \iff b - a > 0$.

(b) $a < b \iff a + c < b + c$.

(c) If $c > 0$, then

$$a < b \iff ac < bc;$$

if $c < 0$, then

$$a < b \iff ac > bc.$$

(d) If $a, b > 0$, then

$$a < b \iff \frac{1}{a} > \frac{1}{b}.$$

(e) If $a, b \geq 0$ and $p > 0$, then

$$a < b \iff a^p < b^p.$$

Corresponding versions of these inequalities exist with *strict* inequalities replaced with *weak* inequalities; that is, with $a < b$ replaced by $a \leq b$.

The **solution set** of an inequality involving an unknown real number x is the set of values of x for which the inequality holds.

Rules for deducing new inequalities from given ones

(a) For all a, b, c in $\mathbb{R}$,

$$a < b \text{ and } b < c \implies a < c. \qquad \text{Transitive Rule}$$

(b) For all a, b, c, d in $\mathbb{R}$, if $a < b$ and $c < d$, then

$$a + c < b + d; \qquad \text{Sum Rule}$$
$$ac < bd \quad \text{(provided that } a, c \geq 0). \qquad \text{Product Rule}$$

Modulus

If $x \in \mathbb{R}$, then the **modulus**, or **absolute value**, of x is

$$|x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases}$$

Thus $|x|$ is the distance from the origin to x, and so

(a) $|x| < a \iff -a < x < a$;

(b) $|x| > a \iff x > a$ or $x < -a$;

(c) the distance on the real line from a to b is $|a - b|$.

5. The Principle of Mathematical Induction

Suppose that $P(n)$, $n = 1, 2, 3, \ldots$, is a sequence of propositions such that

(a) $P(1)$ is true, and

(b) whenever $P(k)$ is true, $P(k+1)$ is also true.

Then $P(n)$ is true, for $n = 1, 2, \ldots$.

Example We wish to show that $2^n \geq n$, for $n = 1, 2, \ldots$.

Let $P(n)$ be the proposition $2^n \geq n$.

(a) $P(1)$ is true, since $2^1 \geq 1$.

(b) Suppose that $P(k)$ is true; that is, $2^k \geq k$. Then

$$2^{k+1} = 2 \cdot 2^k \geq 2k = k + k \geq k + 1,$$

so that $P(k+1)$ is true.

Thus $2^n \geq n$, for $n = 1, 2, \ldots$, by the Principle of Mathematical Induction.

Real functions

1. Definitions

A **real function** f is defined by specifying

(a) two subsets A and B of $\mathbb{R}$, and

(b) a rule which associates with each $x \in A$ a unique $y \in B$.

The sets A and B are called the **domain** and the **codomain** of f, respectively. We write

$$f : A \longrightarrow B.$$

If $f : A \longrightarrow B$ and $x \in A$, then $f(x)$ is called the **image** of x under f, or the **value** of f at x. The image of the function f is

$$f(A) = \{f(x) : x \in A\}.$$

2. Standard functions

Type	Rule	Domain
Polynomial functions	$a_0 + a_1 x + \cdots + a_n x^n$	$\mathbb{R}$
Rational functions	$p(z)/q(z)$, p and q polynomial functions (q not the zero function)	$\mathbb{R} - \{z : q(z) = 0\}$
Trigonometric functions	$\sin x$	$\mathbb{R}$
	$\cos x$	$\mathbb{R}$
	$\tan x$	$\mathbb{R} - \{(n + \frac{1}{2})\pi : n \in \mathbb{Z}\}$
Exponential functions	e^x	$\mathbb{R}$
	a^x, where $a > 0$	$\mathbb{R}$
Natural log functions	$\log_e x$	$\{x : x > 0\}$
Hyperbolic functions	$\sinh x$	$\mathbb{R}$
	$\cosh x$	$\mathbb{R}$
	$\tanh x$	$\mathbb{R}$

Remarks

(a) Definition of e^x:
$$e^x = \sum_{n=0}^{\infty} x^n/n! = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n$$

(b) Definition of $\log_e x$:
$$x = e^y \implies y = \log_e x$$

(c) Definition of a^x (for $a > 0$):
$$a^x = \exp(x \log_e a)$$

(d) Index laws:
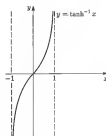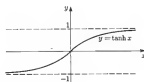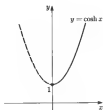$$a^{x+y} = a^x a^y, \quad (a^x)^y = a^{xy}, \quad \sqrt[n]{a^m} = a^{m/n}, \quad a^{-x} = 1/a^x$$
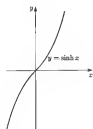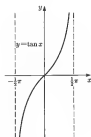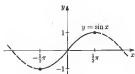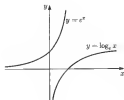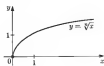
(e) Logarithmic identities:
$$\log_e xy = \log_e x + \log_e y, \quad \log_e(1/x) = -\log_e x$$

3. Trigonometric and hyperbolic identities

$\cos^2 x + \sin^2 x = 1$	$\cosh^2 x - \sinh^2 x = 1$
$1 + \tan^2 x = \sec^2 x$	$1 - \tanh^2 x = \operatorname{sech}^2 x$
$\cot^2 x + 1 = \operatorname{cosec}^2 x$	$\coth^2 x - 1 = \operatorname{cosech}^2 x$
$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$	$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$
$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$	$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$
$\tan(x \pm y) = \dfrac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$	$\tanh(x \pm y) = \dfrac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$
$\sin 2x = 2 \sin x \cos x$	$\sinh 2x = 2 \sinh x \cosh x$
$\cos 2x = \cos^2 x - \sin^2 x$	$\cosh 2x = \cosh^2 x + \sinh^2 x$
$ = 2\cos^2 x - 1$	$ = 2\cosh^2 x - 1$
$ = 1 - 2\sin^2 x$	$ = 1 + 2\sinh^2 x$
$\tan 2x = \dfrac{2\tan x}{1 - \tan^2 x}$	$\tanh 2x = \dfrac{2\tanh x}{1 + \tanh^2 x}$
$\sin(-x) = -\sin x$	$\sinh(-x) = -\sinh x$
$\cos(-x) = \cos x$	$\cosh(-x) = \cosh x$
$\tan(-x) = -\tan x$	$\tanh(-x) = -\tanh x$

4. Graphs of standard functions and their inverses

$y = x^n$

$y = \sqrt[n]{x}$

$y = x^x$
$y = \log_a x$

$y = \sin x$

$y = \cos x$

$y = \tan x$

$y = \sin^{-1} x$

$y = \cos^{-1} x$

$y = \tan^{-1} x$

$y = \sinh x$

$y = \cosh x$

$y = \tanh x$

$y = \sinh^{-1} x$

$y = \cosh^{-1} x$

$y = \tanh^{-1} x$

5. Standard derivatives

Rule of f	Rule of f'	Domain of f'
k	0	$\mathbb{R}$
x^n, $n \in \mathbb{Z} - \{0\}$	nx^{n-1}	$\mathbb{R}$ if $n > 0$; $\mathbb{R} - \{0\}$ if $n < 0$
x^α, $\alpha \in \mathbb{R} - \{0\}$	$\alpha x^{\alpha-1}$	$]0, \infty[$
e^x	e^x	$\mathbb{R}$
a^x, $a > 0$	$a^x \log_e a$	$\mathbb{R}$
$\log_e x$	$1/x$	$]0, \infty[$
$\sin x$	$\cos x$	$\mathbb{R}$
$\cos x$	$-\sin x$	$\mathbb{R}$
$\tan x$	$\sec^2 x$	$\mathbb{R} - \{(n+\frac{1}{2})\pi : n \in \mathbb{Z}\}$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$	$\mathbb{R} - \{n\pi : n \in \mathbb{Z}\}$
$\sec x$	$\sec x \tan x$	$\mathbb{R} - \{(n+\frac{1}{2})\pi : n \in \mathbb{Z}\}$
$\cot x$	$-\operatorname{cosec}^2 x$	$\mathbb{R} - \{n\pi : n \in \mathbb{Z}\}$
$\sin^{-1} x$	$1/\sqrt{1-x^2}$	$]-1, 1[$
$\cos^{-1} x$	$-1/\sqrt{1-x^2}$	$]-1, 1[$
$\tan^{-1} x$	$1/(1+x^2)$	$\mathbb{R}$
$\sinh x$	$\cosh x$	$\mathbb{R}$
$\cosh x$	$\sinh x$	$\mathbb{R}$
$\tanh x$	$\operatorname{sech}^2 x$	$\mathbb{R}$
$\sinh^{-1} x$	$1/\sqrt{1+x^2}$	$\mathbb{R}$
$\cosh^{-1} x$	$1/\sqrt{x^2-1}$	$]1, \infty[$
$\tanh^{-1} x$	$1/(1-x^2)$	$]-1, 1[$

6. Standard primitives

Rule of f	Rule of F ($F' = f$)	Domain of F
x^n, $n \in \mathbb{Z} - \{-1, 0\}$	$x^{n+1}/(n+1)$	$\mathbb{R}$ if $n > 0$; $\mathbb{R} - \{0\}$ if $n < 1$
x^α, $\alpha \in \mathbb{R} - \{-1, 0\}$	$x^{\alpha+1}/(\alpha+1)$	$]0, \infty[$
$1/x$	$\log_e |x|$	$\mathbb{R} - \{0\}$
e^x	e^x	$\mathbb{R}$
a^x, $a > 0$	$a^x / \log_e a$	$\mathbb{R}$
$\log_e x$	$x \log_e x - x$	$]0, \infty[$
$\sin x$	$-\cos x$	$\mathbb{R}$
$\cos x$	$\sin x$	$\mathbb{R}$
$\tan x$	$\log_e |\sec x|$	$\mathbb{R} - \{(n+\frac{1}{2})\pi : n \in \mathbb{Z}\}$
$1/\sqrt{1-x^2}$	$\begin{cases} \sin^{-1} x \\ -\cos^{-1} x \end{cases}$	$]-1, 1[$ $]-1, 1[$
$1/(1+x^2)$	$\tan^{-1} x$	$\mathbb{R}$
$\sinh x$	$\cosh x$	$\mathbb{R}$
$\cosh x$	$\sinh x$	$\mathbb{R}$
$\tanh x$	$\log_e(\cosh x)$	$\mathbb{R}$
$1/\sqrt{1+x^2}$	$\sinh^{-1} x$	$\mathbb{R}$
$1/\sqrt{x^2-1}$	$\cosh^{-1} x$	$]1, \infty[$
$1/(1-x^2)$	$\tanh^{-1} x$	$]-1, 1[$

PART II: UNIT SUMMARIES
Unit A1   Complex Numbers

Section 1: Introducing Complex Numbers

1. A **complex number** z is an expression of the form $x + iy$, where $x, y \in \mathbb{R}$ and i is a symbol with the property that $i^2 = -1$. We write $z = x + iy$ or, equivalently, $z = x + yi$, and say that z is expressed in Cartesian form; x is called the **real part** of z and y the **imaginary part** of z, written $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$.

   Two complex numbers are **equal** if their real parts are equal and their imaginary parts are equal.

   The set of all complex numbers is denoted by $\mathbb{C}$.

2. The binary operations of addition, subtraction and multiplication of complex numbers are denoted by the same symbols as for real numbers and are performed by treating the complex numbers as real expressions involving an algebraic symbol i with the property that $i^2 = -1$.

3. The **negative**, $-z$, of a complex number $z = x + iy$ is

   $$-z = (-x) + i(-y),$$

   usually written $-z = -x - iy$.

4. The **reciprocal**, $1/z$ (or z^{-1}), of a non-zero complex number $z = x + iy$ is

   $$\frac{1}{z} = \frac{x - iy}{x^2 + y^2}.$$

5. The **quotient**, z_1/z_2, of a complex number z_1 by a non-zero complex number z_2 is

   $$\frac{z_1}{z_2} = z_1 \left(\frac{1}{z_2} \right).$$

6. **Strategy for obtaining a quotient**
   To obtain the quotient
   $$\frac{x_1 + iy_1}{x_2 + iy_2},$$
   where $x_2 + iy_2 \neq 0$, in Cartesian form, multiply both numerator and denominator by $x_2 - iy_2$, so that the denominator becomes real.

7. The **complex conjugate**, $\overline{z}$, of a complex number $z = x + iy$ is
   $$\overline{z} = x - iy.$$

8. **Properties of the complex conjugate**
   (a) If z is a complex number, then
   (i) $z + \overline{z} = 2 \operatorname{Re} z$;
   (ii) $z - \overline{z} = 2i \operatorname{Im} z$;
   (iii) $\overline{(\overline{z})} = z$.

   (b) If z_1 and z_2 are complex numbers, then
   (i) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$;
   (ii) $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$;
   (iii) $\overline{z_1 z_2} = \overline{z_1}\, \overline{z_2}$;
   (iv) $\overline{z_1 / z_2} = \overline{z_1} / \overline{z_2}$,   where $z_2 \neq 0$.

9. **Arithmetic in $\mathbb{C}$**

   Addition
   A1 For all z_1, z_2 in $\mathbb{C}$,
   $z_1 + z_2 \in \mathbb{C}$.
   A2 For all z in $\mathbb{C}$,
   $z + 0 = 0 + z = z$.
   A3 For all z in $\mathbb{C}$,
   $z + (-z) = (-z) + z = 0$.
   A4 For all z_1, z_2, z_3 in $\mathbb{C}$,
   $(z_1 + z_2) + z_3$
   $\quad = z_1 + (z_2 + z_3)$.
   A5 For all z_1, z_2 in $\mathbb{C}$,
   $z_1 + z_2 = z_2 + z_1$.

   Multiplication
   M1 For all z_1, z_2 in $\mathbb{C}$,
   $z_1 z_2 \in \mathbb{C}$.
   M2 For all z in $\mathbb{C}$,
   $z1 = 1z = z$.
   M3 For all non-zero z in $\mathbb{C}$,
   $zz^{-1} = z^{-1}z = 1$.
   M4 For all z_1, z_2, z_3 in $\mathbb{C}$,
   $(z_1 z_2) z_3 = z_1 (z_2 z_3)$.
   M5 For all z_1, z_2 in $\mathbb{C}$,
   $z_1 z_2 = z_2 z_1$.
   D For all z_1, z_2, z_3 in $\mathbb{C}$,
   $z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3$.

10. **Binomial Theorem**
    (a) If $z \in \mathbb{C}$ and $n \in \mathbb{N}$, then
    $$(1+z)^n = \sum_{k=0}^{n} \binom{n}{k} z^k$$
    $$= 1 + nz + \frac{n(n-1)}{2!}z^2 + \cdots + z^n.$$
    (b) If z_1, $z_2 \in \mathbb{C}$ and $n \in \mathbb{N}$, then
    $$(z_1 + z_2)^n = \sum_{k=0}^{n} \binom{n}{k} z_1^{n-k} z_2^k$$
    $$= z_1^n + n z_1^{n-1} z_2 + \frac{n(n-1)}{2!} z_1^{n-2} z_2^2 + \cdots + z_2^n.$$

11. **Geometric Series Identity**
    (a) If $z \in \mathbb{C}$ and $n \in \mathbb{N}$, then
    $$1 - z^n = (1 - z)\left(1 + z + z^2 + \cdots + z^{n-1}\right).$$
    (b) If z_1, $z_2 \in \mathbb{C}$ and $n \in \mathbb{N}$, then
    $$z_1^n - z_2^n = (z_1 - z_2)\left(z_1^{n-1} + z_1^{n-2} z_2 + z_1^{n-3} z_2^2 + \cdots + z_2^{n-1}\right).$$

Section 2: The Complex Plane

1. The complex number $z = x + iy$ is represented by the ordered pair (x, y) in $\mathbb{R}^2$, called the complex plane. Complex numbers can be thought of as based vectors; $z = x + iy$ corresponds to the vector from $(0,0)$ to (x, y). The sum and difference of two complex numbers z_1 and z_2 satisfy the parallelogram law for vectors.

2. The modulus, or absolute value, of $z = x + iy$ is the distance from 0 to z; it is denoted by $|z|$. Thus
$$|z| = |x + iy| = \sqrt{x^2 + y^2}.$$

3. $|z_1 - z_2|$ is the distance from z_1 to z_2.
$|z_1 + z_2|$ is the distance from z_1 to $-z_2$.

4. Properties of the modulus
   (a) $|z| \geq 0$, with equality if and only if $z = 0$;
   (b) $|\overline{z}| = |z|$ and $|-z| = |z|$;
   (c) $|z|^2 = z\overline{z}$;
   (d) $|z_1 - z_2| = |z_2 - z_1|$;
   (e) $|z_1 z_2| = |z_1||z_2|$
       and
       $|z_1/z_2| = |z_1|/|z_2|$, where $z_2 \neq 0$.

5. An argument of $z = x + iy$, $z \neq 0$, is an angle θ (measured in radians) such that
$$\cos \theta = \frac{x}{|z|} \quad \text{and} \quad \sin \theta = \frac{y}{|z|}.$$
No argument is assigned to $z = 0$.
Each $z \neq 0$ has infinitely many arguments. If θ is an argument of z, then the arguments of z are given by $\theta + 2n\pi$, $n \in \mathbb{Z}$.

6. The ordered pair (r, θ), where r is the modulus of a non-zero complex number z and θ is an argument of z, are called polar coordinates of z. The expression
$$z = r(\cos \theta + i \sin \theta)$$
is called a representation of z in polar form.

7. The principal argument of a non-zero complex number z is the unique argument θ satisfying $-\pi < \theta \leq \pi$; it is denoted by
$$\theta = \text{Arg } z.$$

8. Strategy for determining principal arguments
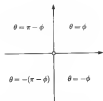To determine the principal argument θ of a non-zero complex number $z = x + iy$:
Case (i) If z lies on one of the axes, then θ is evident.
Case (ii) If z does not lie on one of the axes, then
   (a) decide in which quadrant z lies (by plotting if necessary) and then calculate the angle
$$\phi = \tan^{-1}(|y|/|x|);$$
   (b) obtain θ in terms of ϕ by using the appropriate formula in the following diagram.

$\theta = \pi - \phi$ $\theta = \phi$

$\theta = -(\pi - \phi)$ $\theta = -\phi$

9. If z_1 and z_2 are non-zero with
$z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$ and $z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$,
then
$$z_1 z_2 = r_1 r_2(\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)).$$
The geometric effect on z_1 of multiplying it by z_2 is to scale z_1 by the factor $|z_2|$ and rotate it about 0 through the angle Arg z_2 (this rotation is anticlockwise if Arg $z_2 > 0$, clockwise if Arg $z_2 < 0$).

10. If z_1 and z_2 are (non-zero) complex numbers, then
$$\text{Arg}(z_1 z_2) = \text{Arg } z_1 + \text{Arg } z_2 + 2n\pi,$$
where n is -1, 0 or 1 according as Arg $z_1 + $ Arg z_2 is greater than π, lies in the interval $]-\pi, \pi]$ or is less than or equal to $-\pi$.

11. If z_1 and z_2 are non-zero with
$z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$ and $z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$,
then
$$\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)).$$
The geometric effect on z_1 of dividing it by z_2 is to scale z_1 by the factor $1/|z_2|$ and rotate it about 0 through the angle Arg z_2 (this rotation is clockwise if Arg $z_2 > 0$, anticlockwise if Arg $z_2 < 0$).

12. If z is non-zero and $z = r(\cos \theta + i \sin \theta)$, then
$$z^{-1} = \frac{1}{r}(\cos(-\theta) + i \sin(-\theta)).$$

13. de Moivre's Theorem
If n is an integer and θ is a real number, then
$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta.$$

Section 3: Solving Equations with Complex Numbers

1. Suppose that w is a non-zero complex number and $n \geq 2$. Each solution of $z^n = w$ is called an nth root of w.

2. Let
$$w = \rho(\cos\phi + i\sin\phi)$$
be a non-zero complex number in polar form. Then w has exactly n nth roots, given by
$$z_k = \rho^{1/n}\left(\cos\left(\frac{\phi}{n} + k\frac{2\pi}{n}\right) + i\sin\left(\frac{\phi}{n} + k\frac{2\pi}{n}\right)\right),$$
$$k = 0, 1, \ldots, n-1.$$
These nth roots are the vertices of an n-sided regular polygon inscribed in a circle of radius $\rho^{1/n}$ centred at 0.

3. If $w = \rho(\cos\phi + i\sin\phi)$, where ϕ is the principal argument of w, then
$$z_0 = \rho^{1/n}\left(\cos\frac{\phi}{n} + i\sin\frac{\phi}{n}\right)$$
is called the principal nth root of w, denoted by $\sqrt[n]{w}$ or $w^{1/n}$.

4. By an nth root of unity we mean an nth root of 1. These n roots are given by
$$z_k = \cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n}, \quad k = 0, 1, 2, \ldots, n-1.$$

5. Strategy for finding nth roots
To find the n nth roots $z_0, z_1, \ldots, z_{n-1}$ of a non-zero complex number:
  (a) express w in polar form, with modulus ρ and argument ϕ;
  (b) substitute the values of ρ and ϕ in the formula
$$z_k = \rho^{1/n}\left(\cos\left(\frac{\phi}{n} + k\frac{2\pi}{n}\right) + i\sin\left(\frac{\phi}{n} + k\frac{2\pi}{n}\right)\right),$$
$$k = 0, 1, \ldots, n-1;$$
  (c) if required, convert the roots to Cartesian form.

6. The two roots of $az^2 + bz + c = 0$, where a, b, c in $\mathbb{C}$ and $a \neq 0$, are
$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$
In certain cases a polynomial equation of degree n can be solved by substitution to obtain a quadratic equation.

Section 4: Sets of Complex Numbers

1. Inequalities between complex numbers have no meaning, unless the complex numbers are real. Since $|z|$, $\operatorname{Re} z$, $\operatorname{Im} z$ and $\operatorname{Arg} z$ are real, we can form inequalities involving these.

2. The following subsets of $\mathbb{C}$ are commonly used.
  (a) Open half-plane $\{z : a\operatorname{Re} z + b\operatorname{Im} z \geq c\}$, closed half-plane $\{z : a\operatorname{Re} z + b\operatorname{Im} z \geq c\}$, where $a, b, c \in \mathbb{R}$ and a, b are not both zero.
  (b) Open disc $\{z : |z - \alpha| < r\}$, closed disc $\{z : |z - \alpha| \leq r\}$, where $r > 0$.
  (c) Open annulus $\{z : r_1 < |z - \alpha| < r_2\}$, closed annulus $\{z : r_1 \leq |z - \alpha| \leq r_2\}$, where $r_2 \geq r_1 \geq 0$.
  (d) Ray (half-line) $\{z : \operatorname{Arg} z = \theta\}$, where $\theta \in]-\pi, \pi]$.
  (e) Open sector $\{z : \alpha < \operatorname{Arg}(z - \alpha) < b\}$, where $\alpha \in \mathbb{C}$ and $-\pi \leq a < b \leq \pi$.
    A special case is the cut plane
$$\{z : |\operatorname{Arg} z| < \pi\} = \mathbb{C} - \{z \in \mathbb{R} : z \leq 0\}$$

Section 5: Proving Inequalities

1. $|\operatorname{Re} z| \leq |z|$ and $|\operatorname{Im} z| \leq |z|$.

2. Triangle Inequality
If z_1, $z_2 \in \mathbb{C}$, then
  (a) $|z_1 + z_2| \leq |z_1| + |z_2|$ (usual form);
  (b) $|z_1 - z_2| \geq \big||z_1| - |z_2|\big|$ (backwards form).

3. If z, z_1, z_2, $\ldots$, $z_n \in \mathbb{C}$, then
  (a) $|z| \leq |\operatorname{Re} z| + |\operatorname{Im} z|$;
  (b) $|z_1 - z_2| \leq |z_1| + |z_2|$;
  (c) $|z_1 + z_2| \geq \big||z_1| - |z_2|\big|$;
  (d) $|z_1 \pm z_2 \pm \cdots \pm z_n| \leq |z_1| + |z_2| + \cdots + |z_n|$;
  (e) $|z_1 \pm z_2 \pm \cdots \pm z_n| \geq |z_1| - |z_2| - \cdots - |z_n|$.

Unit A2 Complex Functions

Section 1: What is a Complex Function?

1. A **complex function** f is defined by specifying

   (a) two sets A and B in the complex plane $\mathbb{C}$, and

   (b) a rule which associates with each number z in A a unique number w in B; we write $w = f(z)$.

   The set A is called the **domain** of the function f and B is called the **codomain** of f. The number w is called the **image** of z under f, or the **value of f at z**.

2. **Convention** When a function f is specified *just* by a rule, then its domain is understood to be the set of all complex numbers to which the rule is applicable, and the codomain is $\mathbb{C}$.

3. Given a function $f : A \longrightarrow B$, the **image** of f is
   $$f(A) = \{f(z) : z \in A\}.$$
   If $f(A) = B$, then f is called **onto**.

4. Let f be **real-valued**.
   If $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$, then f is a **real function**.

5. Let $f : A \longrightarrow \mathbb{C}$ and $g : B \longrightarrow \mathbb{C}$ be functions. Then
   the **sum** $f + g$ is the function with domain $A \cap B$ and rule
   $$(f + g)(z) = f(z) + g(z);$$
   the **multiple** λf, where $\lambda \in \mathbb{C}$, is the function with domain A and rule
   $$(\lambda f)(z) = \lambda f(z);$$
   the **product** fg is the function with domain $A \cap B$ and rule
   $$(fg)(z) = f(z)g(z);$$
   the **quotient** f/g is the function with domain $A \cap B - \{z : g(z) = 0\}$ and rule
   $$(f/g)(z) = f(z)/g(z).$$

6. A **polynomial function** of degree n is defined by
   $$p(z) = a_0 + a_1 z + \cdots + a_n z^n,$$
   where $a_0, \ldots, a_n \in \mathbb{C}$, and $a_n \neq 0$.
   A **rational function** r is the quotient of two polynomial functions p and q: $r(z) = \dfrac{p(z)}{q(z)}$.

7. Let $f : A \longrightarrow \mathbb{C}$ and $g : B \longrightarrow \mathbb{C}$ be complex functions. Then the **composite** function $g \circ f$ has domain
   $$\{z \in A : f(z) \in B\}$$
   and rule
   $$(g \circ f)(z) = g(f(z)).$$

8. The function $f : A \longrightarrow B$ is **one-one** if the images under f of distinct points in A are also distinct; that is,
   $$\text{if } z_1, z_2 \in A \text{ and } z_1 \neq z_2, \text{ then } f(z_1) \neq f(z_2).$$
   Equivalently, if $w \in f(A)$ then there is a **unique** $z \in A$ such that $f(z) = w$.
   A function which is not one-one is **many-one**.

9. Let $f : A \longrightarrow B$ be a one-one function. Then the **inverse function**, f^{-1}, of f has domain $f(A)$ and rule
   $$f^{-1}(w) = z, \quad \text{where } w = f(z).$$

10. **Strategy** for proving that a function f has an inverse function
    EITHER   prove that f is one-one directly by showing that
    $$\text{if } z_1 \neq z_2, \text{ then } f(z_1) \neq f(z_2)$$
    OR   determine the image $f(A)$ and show that for each $w \in f(A)$ there is a unique $z \in A$ such that
    $$f(z) = w.$$

11. Reducing the domain of a function (but leaving the rule unchanged) gives a **restriction** of the function.

Section 2: Special Types of Complex Function

1. The functions $\operatorname{Re} f : z \longmapsto \operatorname{Re}(f(z))$ and $\operatorname{Im} f : z \longmapsto \operatorname{Im}(f(z))$ are called the **real** and the **imaginary parts** of f. $\operatorname{Re} f$ and $\operatorname{Im} f$ are real-valued functions with the same domain as f.

2. A **path** is a subset Γ of $\mathbb{C}$ which is the image of an associated continuous function $\gamma : I \longrightarrow \mathbb{C}$, where I is a real interval.
   The function γ is called a **parametrization**. If
   $$\gamma(t) = \phi(t) + i\psi(t) \quad (t \in I)$$
   where ϕ and ψ are real functions, then the equations
   $$z = \phi(t), \quad y = \psi(t) \quad (t \in I),$$
   are called **parametric equations** of γ.
   If I is the closed interval $[a, b]$, then $\gamma(a)$ and $\gamma(b)$ are called the **initial point** and **final point** of Γ, respectively.
   It may be possible to obtain the equation of Γ in terms of x and y alone by eliminating t from the equations $z = \phi(t)$ and $y = \psi(t)$.

3. Standard parametrizations

Set	Standard parametrization
Line through points α and β.	$\gamma(t) = (1-t)\alpha + t\beta$ $(t \in \mathbb{R})$
Line segment between points α and β.	$\gamma(t) = (1-t)\alpha + t\beta$ $(t \in [0,1])$
Circle with centre α, radius r: $|z - \alpha| = r$.	$\gamma(t) = \alpha + r(\cos t + i\sin t)$ $(t \in [0, 2\pi])$
Arc of circle with centre α, radius r.	$\gamma(t) = \alpha + r(\cos t + i\sin t)$ $(t \in [t_1, t_2])$
Ellipse in standard form: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.	$\gamma(t) = a\cos t + ib\sin t$ $(t \in [0, 2\pi])$
Parabola in standard form: $y^2 = 4ax$.	$\gamma(t) = at^2 + 2iat$ $(t \in \mathbb{R})$
Right half of hyperbola in standard form: $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$.	$\gamma(t) = a\cosh t + ib\sinh t$ $(t \in \mathbb{R})$

4. Given a function $f : A \longrightarrow B$ and a subset S of A, the image under f of S is
$$f(S) = \{f(s) : s \in S\}.$$

5. If f is a continuous function and Γ is a path in the domain of f, with parametrization γ, then $f(\Gamma)$ is called the image path; $f(\Gamma)$ has parametrization $f \circ \gamma$.

6. Strategy for determining an image path

Let the parametrization of the path Γ be
$$\gamma(t) = \phi(t) + i\psi(t) \quad (t \in I).$$
Then the image path $f(\Gamma)$ is found

EITHER

by using the geometric properties of the continuous function f

OR

by substituting $x = \phi(t)$, $y = \psi(t)$ into the equation
$$u + iv = f(x + iy)$$
and then, by equating real parts and imaginary parts, obtaining expressions for u and v in terms of t. (These expressions are the parametric equations of the image path $f(\Gamma)$, associated with the parametrization $f \circ \gamma$.)

Section 3: Images of Grids

A **Cartesian grid** consists of lines of the form $x = a$ and $y = b$, usually evenly spaced in both directions.

A **polar grid** consists of circles centred at 0 and rays emerging from 0. Each of the circles has an equation of the form $r = a$, where a is a positive constant, and each of the rays has an equation of the form $\theta = b$, where b is a constant in the interval $[-\pi, \pi]$.

Using the Strategy in Section 2 for determining an image path $f(\Gamma)$, we can obtain the images of Cartesian and polar grids.

Section 4: Exponential, Trigonometric and Hyperbolic Functions

1. For all $z = x + iy$ in $\mathbb{C}$,
$$e^z = e^x(\cos y + i\sin y).$$
The function
$$z \longmapsto e^z \quad (z \in \mathbb{C})$$
is called the **exponential function** and is denoted by exp.

If z is real, $z = x + 0i$, then
$$e^z = e^x(\cos 0 + i\sin 0) = e^x.$$
If z is imaginary, $z = 0 + iy$, then
$$e^{iz} = \cos y + i\sin y.$$

2. Exponential identities

(a) $e^{z_1 + z_2} = e^{z_1} e^{z_2}$

(b) $|e^z| = e^{\operatorname{Re} z}$

(c) $e^{-z} = 1/e^z$

(d) $e^{z + 2\pi i} = e^z$

3. The geometric nature of exp

(a) For all $n \in \mathbb{Z}$, $e^{z + 2n\pi i} = e^z$, so each of the points $z + 2n\pi i$, $n \in \mathbb{Z}$, has the same image.

(b) The line $x = a$ is mapped to the path with parametric equations
$$u = e^a \cos t, \quad v = e^a \sin t \quad (t \in \mathbb{R}).$$
This is the circle with radius e^a and centre 0.

(c) The line $y = b$ is mapped to the path with parametric equations
$$u = e^t \cos b, \quad v = e^t \sin b \quad (t \in \mathbb{R}).$$
This is the ray from 0 (excluded) through $\cos b + i\sin b$.

(d) The strip $\{x + iy : -\pi < y \leq \pi\}$ is mapped to $\mathbb{C} - \{0\}$.

4. Trigonometric functions

The functions cos, sin, tan, sec, cot and cosec are defined as follows:

$$\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right), \quad \sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right),$$

$$\tan z = \frac{\sin z}{\cos z}, \qquad \sec z = \frac{1}{\cos z},$$

$$\cot z = \frac{\cos z}{\sin z}, \qquad \operatorname{cosec} z = \frac{1}{\sin z}.$$

Both cos and sin have domain $\mathbb{C}$.

Since $\{z : \cos z = 0\} = \{(n + \frac{1}{2})\pi : n \in \mathbb{Z}\}$, both tan and sec have domain $\mathbb{C} - \{(n + \frac{1}{2})\pi : n \in \mathbb{Z}\}$.

Since $\{z : \sin z = 0\} = \{n\pi : n \in \mathbb{Z}\}$, both cot and cosec have domain $\mathbb{C} - \{n\pi : n \in \mathbb{Z}\}$.

5. All the identities satisfied by the real trigonometric functions also hold for the complex trigonometric functions.

6. Hyperbolic functions

The functions cosh, sinh, tanh, sech, coth and cosech are defined as follows:

$$\cosh z = \frac{1}{2}\left(e^z + e^{-z}\right), \quad \sinh z = \frac{1}{2}\left(e^z - e^{-z}\right),$$

$$\tanh z = \frac{\sinh z}{\cosh z}, \qquad \operatorname{sech} z = \frac{1}{\cosh z},$$

$$\coth z = \frac{\cosh z}{\sinh z}, \qquad \operatorname{cosech} z = \frac{1}{\sinh z}.$$

Both cosh and sinh have domain $\mathbb{C}$.

Since $\{z : \cosh z = 0\} = \{(n + \frac{1}{2})\pi i : n \in \mathbb{Z}\}$, both tanh and sech have domain $\mathbb{C} - \{(n + \frac{1}{2})\pi i : n \in \mathbb{Z}\}$.

Since $\{z : \sinh z = 0\} = \{n\pi i : n \in \mathbb{Z}\}$, both coth and cosech have domain $\mathbb{C} - \{n\pi i : n \in \mathbb{Z}\}$.

7. For all z in $\mathbb{C}$,

$$\sin(iz) = i \sinh z \quad \text{and} \quad \cos(iz) = \cosh z.$$

8. All the identities satisfied by the real hyperbolic functions also hold for the complex hyperbolic functions.

2. Logarithmic identities

(a) $\operatorname{Log}(z_1 z_2) = \operatorname{Log} z_1 + \operatorname{Log} z_2$,
    if $\operatorname{Arg} z_1, \operatorname{Arg} z_2 \in \left]-\frac{1}{2}\pi, \frac{1}{2}\pi\right]$.

(b) $\operatorname{Log}(1/z) = -\operatorname{Log} z$, if $\operatorname{Arg} z \in \left]-\pi, \pi\right[$.

Part (a) holds in the following form for any values in the domain, $\mathbb{C} - \{0\}$, of Log:

$$\operatorname{Log} z_1 z_2 = \operatorname{Log} z_1 + \operatorname{Log} z_2 + 2n\pi i,$$

where n is -1, 0 or 1 according as $\operatorname{Arg} z_1 + \operatorname{Arg} z_2$ is greater than π, lies in the interval $]-\pi, \pi]$, or is less than or equal to $-\pi$.

(c) For $z, \alpha \in \mathbb{C}$, with $z \neq 0$, the principal αth power of z is

$$z^\alpha = \exp(\alpha \operatorname{Log} z).$$

This definition agrees with the usual meaning of z^n, for $\alpha = n$ or $\alpha = 1/n$, where $n \in \mathbb{N}$.

The function $z \mapsto z^\alpha$ is called the principal αth power function.

Section 5: Logarithms and Powers

1. For $z \in \mathbb{C} - \{0\}$, the principal logarithm of z is

$$\operatorname{Log} z = \log_e |z| + i \operatorname{Arg} z.$$

The corresponding principal logarithm function is called Log. It is the inverse of

$$f(z) = e^z \quad (z \in \{x + iy : -\pi < y \leq \pi\}),$$

and satisfies

$$e^{\operatorname{Log} z} = z, \quad \text{for } z \in \mathbb{C} - \{0\},$$

and

$$\operatorname{Log}(e^z) = z, \quad \text{for } z \in \{x + iy : -\pi < y \leq \pi\}.$$

Unit A3 Continuity

Section 1: Sequences

1. A (**complex**) **sequence** is an unending list of complex numbers

$$z_1, z_2, z_3, \ldots.$$

The complex number z_n is called the **nth term** of the sequence, and the sequence is denoted by $\{z_n\}$.

2. The sequence $\{z_n\}$ is **convergent** with limit α, or **converges to** α, or **tends to** α, if for each positive number ε, there is an integer N such that

$$|z_n - \alpha| < \varepsilon, \quad \text{for all } n > N.$$

If $\{z_n\}$ converges to α, then we write

EITHER $\lim\limits_{n \to \infty} z_n = \alpha$

OR $z_n \to \alpha$ as $n \to \infty$.

If the limit α is 0, then $\{z_n\}$ is called a **null sequence**.

3. (a) The sequence $\{z_n\}$ converges to α if and only if the sequence $\{z_n - \alpha\}$ is null.

(b) If a sequence $\{z_n\}$ is convergent, then it has a unique limit.

(c) A constant sequence

$$z_n = \alpha, \quad n = 1, 2, \ldots,$$

is convergent with limit α.

(d) If a sequence converges to α, then this remains true if we *add, delete, or alter a finite number of terms*.

4. Squeeze Rule

If $\{a_n\}$ is a real null sequence of non-negative terms, and if

$$|z_n| \le a_n, \quad \text{for } n = 1, 2, \ldots,$$

then $\{z_n\}$ is a null sequence.

5. Basic null sequences

The following sequences are null:

(a) $\{1/n^p\}$, for $p > 0$;

(b) $\{\alpha^n\}$, for $|\alpha| < 1$.

6. Combination Rules

If $\lim\limits_{n \to \infty} z_n = \alpha$ and $\lim\limits_{n \to \infty} w_n = \beta$, then

Sum Rule $\lim\limits_{n \to \infty} (z_n + w_n) = \alpha + \beta$;

Multiple Rule $\lim\limits_{n \to \infty} (\lambda z_n) = \lambda \alpha$, where $\lambda \in \mathbb{C}$;

Product Rule $\lim\limits_{n \to \infty} (z_n w_n) = \alpha \beta$;

Quotient Rule $\lim\limits_{n \to \infty} \left(\dfrac{z_n}{w_n} \right) = \dfrac{\alpha}{\beta}$,

provided that $\beta \ne 0$.

7. If $\lim\limits_{n \to \infty} z_n = \alpha$, then

(a) $\lim\limits_{n \to \infty} |z_n| = |\alpha|$;

(b) $\lim\limits_{n \to \infty} \overline{z_n} = \overline{\alpha}$;

(c) $\lim\limits_{n \to \infty} \operatorname{Re} z_n = \operatorname{Re} \alpha$;

(d) $\lim\limits_{n \to \infty} \operatorname{Im} z_n = \operatorname{Im} \alpha$.

8. A sequence which is not convergent is **divergent**.

9. The sequence $\{z_n\}$ **tends to infinity** if, for each positive number M, there is an integer N such that

$$|z_n| > M, \quad \text{for all } n > N.$$

In this case we write

$$z_n \to \infty \text{ as } n \to \infty.$$

10. Reciprocal Rule

Let $\{z_n\}$ be a sequence. Then

$$z_n \to \infty \text{ as } n \to \infty$$

if and only if

$\{1/z_n\}$ is a null sequence.

11. Let $\{n_k\}$ be a sequence of positive integers which are strictly increasing; that is,

$$n_1 < n_2 < n_3 < \ldots.$$

Then the sequence $\{z_{n_k}\}$ is a **subsequence** of the sequence $\{z_n\}$.

In particular, $\{z_{2k}\}$ is the **even subsequence** and $\{z_{2k-1}\}$ is the **odd subsequence**.

12. Subsequence Rules

(a) **First Subsequence Rule** The sequence $\{z_n\}$ is divergent if $\{z_n\}$ has two convergent subsequences with different limits.

(b) **Second Subsequence Rule** The sequence $\{z_n\}$ is divergent if $\{z_n\}$ has a subsequence which tends to infinity.

13. (a) If $|\alpha| > 1$, then the sequence $\{\alpha^n\}$ tends to infinity.

(b) If $|\alpha| = 1$ and $\alpha \ne 1$, then the sequence $\{\alpha^n\}$ is divergent.

14. If $\{z_n\}$ is a convergent sequence, then there is a positive number M such that

$$|z_n| \le M, \quad \text{for } n = 1, 2, \ldots.$$

In this case, we say that $\{z_n\}$ is **bounded**.

Section 2: Continuous Functions

1. **Continuity: sequential definition**

   Let $f : A \to \mathbb{C}$ and $\alpha \in A$. Then f is **continuous at** α if, for each sequence $\{x_n\}$ in A such that $x_n \to \alpha$, we have
   $$f(x_n) \to f(\alpha);$$
   that is,
   $$x_n \to \alpha \implies f(x_n) \to f(\alpha).$$
   If f is continuous at each α in A, then we say that f is continuous on A.

   If f is not continuous at α, then we say that f is **discontinuous at** α.

2. **Continuity: ε-δ definition**

   Let $f : A \to \mathbb{C}$ and $\alpha \in A$. Then f is continuous at α if, for each $\varepsilon > 0$, there is $\delta > 0$ such that
   $$z \in A, \ |z - \alpha| < \delta \implies |f(z) - f(\alpha)| < \varepsilon.$$

3. The ε-δ definition of continuity is equivalent to the sequential definition of continuity.

4. **Basic continuous functions**

   (a) polynomial and rational functions;

   (b) $f(z) = |z|, \ \overline{z}, \ \mathrm{Re} \, z, \ \mathrm{Im} \, z$;

   (c) $f(z) = e^z$;

   (d) trigonometric and hyperbolic functions;

   (e) $f(z) = \mathrm{Arg} \, z, \ \mathrm{Log} \, z, \ z^n$, on $\mathbb{C} - \{x \in \mathbb{R} : x \leq 0\}$.

5. **Combination Rules**

   If the functions f and g are continuous at α, then so are

Sum Rule	$f + g$;
Multiple Rule	λf, for $\lambda \in \mathbb{C}$;
Product Rule	fg;
Quotient Rule	f/g, provided that $g(\alpha) \neq 0$.

6. **Composition Rule**

   If the function f is continuous at α, and if the function g is continuous at $f(\alpha)$, then $g \circ f$ is continuous at α.

7. **Restriction Rule**

   If the function f has domain A, the function g has domain B and

   1. f is the restriction of g to A;

   2. g is continuous at $\alpha \in A$,

   then f is continuous at α.

Section 3: Limits of Functions

1. The point α is a **limit point** of a set A if there is a sequence $\{x_n\}$ such that
   $$x_n \in A - \{\alpha\}, \quad \text{for } n = 1, 2, \ldots,$$
   and
   $$\lim_{n \to \infty} x_n = \alpha.$$

2. Let f be a function with domain A, and suppose that α is a limit point of A. Then the function f has limit β as x tends to α if for each sequence $\{x_n\}$ in $A - \{\alpha\}$ such that $x_n \to \alpha$,
   $$f(x_n) \to \beta.$$
   In this case we write
   $$\text{EITHER} \quad \lim_{x \to \alpha} f(x) = \beta,$$
   $$\text{OR} \quad f(x) \to \beta \text{ as } x \to \alpha.$$

3. **Strategy for proving that a limit does not exist**

   To prove that $\lim_{x \to \alpha} f(x)$ does not exist, where α is a limit point of the domain A of the function f:

   EITHER

   (a) find two sequences $\{x_n\}$ and $\{x'_n\}$ in $A - \{\alpha\}$ which both tend to α, such that the sequences $\{f(x_n)\}$ and $\{f(x'_n)\}$ have different limits;

   OR

   (b) find a sequence $\{x_n\}$ in $A - \{\alpha\}$ which tends to α, such that the sequence $\{f(x_n)\}$ tends to infinity.

4. Let f be a function with domain A and suppose that $\alpha \in A$ is a limit point of A. Then
   $$f \text{ is continuous at } \alpha \iff \lim_{x \to \alpha} f(x) = f(\alpha).$$

5. Let f be a function with domain A and suppose that α is a limit point of A. Then the function f has limit β as x tends to α if for each positive ε, there is a positive δ such that
   $$|f(x) - \beta| < \varepsilon, \quad \text{for all } x \in A - \{\alpha\} \text{ with } |x - \alpha| < \delta.$$

6. **Combination Rules**

   Let f and g be functions with domains A and B, respectively, and suppose that α is a limit point of $A \cap B$. If
   $$\lim_{x \to \alpha} f(x) = \beta \quad \text{and} \quad \lim_{x \to \alpha} g(x) = \gamma,$$
   then

Sum Rule	$\lim_{x \to \alpha}(f(x) + g(x)) = \beta + \gamma$;
Multiple Rule	$\lim_{x \to \alpha}(\lambda f(x)) = \lambda \beta$, for $\lambda \in \mathbb{C}$;
Product Rule	$\lim_{x \to \alpha}(f(x)g(x)) = \beta \gamma$;
Quotient Rule	$\lim_{x \to \alpha}(f(x)/g(x)) = \beta/\gamma$, provided that $\gamma \neq 0$.

Section 4: Regions

1. A set A in C is open if each point α in A is the centre of some open disc lying entirely in A.

2. Combination Rules

If A_1 and A_2 are open sets, then so are:

(a) $A_1 \cup A_2$ and

(b) $A_1 \cap A_2$.

This rule extends to n open sets $A_1, A_2, \ldots, A_n$:
$$A_1 \cup A_2 \cup \ldots \cup A_n \quad \text{and} \quad A_1 \cap A_2 \cap \ldots \cap A_n$$
are open sets.

3. A set A in C is (pathwise) connected if each pair of points α, β in A can be joined by a path lying entirely in A.

4. Connected sets in C having any two points α and β can be joined by a line segment are called convex.

5. Let f be a continuous function whose domain A is connected. Then the image $f(A)$ is also connected.

6. A region is a non-empty, connected, open set.

7. Basic regions

(a) any open disc;

(b) any open half-plane;

(c) the complement of any closed disc;

(d) any open annulus;

(e) any open rectangle;

(f) any open sector (including cut planes);

(g) the set C itself.

8. If $\mathcal{R}$ is a region and $\alpha_0 \in \mathcal{R}$, then $\mathcal{R} - \{\alpha_0\}$ is also a region.

Section 5: The Extreme Value Theorem

1. A set E in C is closed if its complement $C - E$ is open.

2. If E is a closed set and $\{z_n\}$ is a sequence in E with limit α, then $\alpha \in E$.

3. Combination Rules

If E_1 and E_2 are closed sets, then so are:

(a) $E_1 \cup E_2$ and

(b) $E_1 \cap E_2$.

This rule extends to n closed sets $E_1, E_2, \ldots, E_n$:
$$E_1 \cup E_2 \cup \ldots \cup E_n \quad \text{and} \quad E_1 \cap E_2 \cap \ldots \cap E_n$$
are closed sets.

4. A set E in C is bounded if it is contained in some closed disc.

A set is unbounded if it is not bounded.

5. A set E in C is compact if E is closed and bounded.

6. Extreme Value Theorem

Let the function f be continuous on a compact set E. Then there are numbers α, β in E such that
$$|f(\beta)| \leq |f(z)| \leq |f(\alpha)|, \quad \text{for all } z \in E.$$

7. Boundedness Theorem

Let the function f be continuous on a compact set E. Then there is a number M such that
$$|f(z)| \leq M, \quad \text{for all } z \in E.$$

8. If the function f is continuous on a compact set E, then $f(E)$ is compact.

9. Let A be a subset of C and let $\alpha \in \mathbb{C}$. Then

(a) α is an interior point of A if there is an open disc centred at α which lies entirely in A;

(b) α is an exterior point of A if there is an open disc centred at α which lies entirely outside A.

The set of interior points of A forms the interior of A, written int A, and the set of exterior points of A forms the exterior of A, written ext A.

10. Let A be a subset of C and let $\alpha \in \mathbb{C}$. Then α is a boundary point of A if each open disc centred at α contains at least one point of A and at least one point of $\mathbb{C} - A$.

The set of boundary points of A forms the boundary of A, written ∂A.

The sets int A, ext A and ∂A are disjoint and
$$\partial A = \mathbb{C} - (\text{int } A \cup \text{ext } A).$$

Also int A and ext A are open, whereas ∂A is closed.

11. Nested Rectangles Theorem

Let $R_n, n = 0, 1, 2, \ldots$, be a sequence of closed rectangles with sides parallel to the axes, and with diagonals of lengths $s_n, n = 0, 1, 2, \ldots$, such that

1. $R_0 \supseteq R_1 \supseteq R_2 \supseteq \ldots$, and

2. $\lim_{n \to \infty} s_n = 0$.

Then there is a unique complex number α which lies in all of the rectangles R_n. Moreover, for each positive number ε, there is an integer N such that
$$R_n \subseteq \{z : |z - \alpha| < \varepsilon\}, \quad \text{for all } n > N.$$

Unit A4   Differentiation

Section 1: Derivatives of Complex Functions

1. Let f be a complex function whose domain contains the point α. Then the derivative of f at α is

$$\lim_{z \to \alpha} \frac{f(z) - f(\alpha)}{z - \alpha} \quad \left(\text{or} \quad \lim_{h \to 0} \frac{f(\alpha + h) - f(\alpha)}{h} \right),$$

provided that this limit exists. If the limit does exist, then f is differentiable at α. If f is differentiable at every point of a set A, then f is differentiable on A. A function is differentiable if it is differentiable on its domain.

The derivative of f at α is denoted by $f'(\alpha)$, and the function

$$f' : z \longmapsto f'(z)$$

is called the derivative of f. The domain of f' is the set of all complex numbers at which f is differentiable.

2. A function is entire if it is differentiable on the whole of $\mathbb{C}$.

3. If a function f is differentiable on a region $\mathcal{R}$, then f is said to be analytic on $\mathcal{R}$. If the domain of f is a region and if f is differentiable on this domain, then f is said to be analytic. A function f is analytic at a point α if it is differentiable on a region containing α.

4. If the complex function f is differentiable at α, then f is continuous at α.

5. **Linear Approximation Theorem**
   If the complex function f is differentiable at α, then f may be approximated near α by a linear polynomial. More precisely,

   $$f(z) = f(\alpha) + (z - \alpha)f'(\alpha) + e(z),$$

   where e is an 'error' function satisfying
   $e(z)/(z - \alpha) \to 0$ as $z \to \alpha$.

6. **Combination Rules**
   Let f and g be complex functions with domains A and B, respectively, and let α be a limit point of $A \cap B$. If f and g are differentiable at α, then

   Sum Rule $\quad (f + g)'(\alpha) = f'(\alpha) + g'(\alpha)$;

   Multiple Rule $\quad (\lambda f)'(\alpha) = \lambda f'(\alpha)$;

   Product Rule $\quad (fg)'(\alpha) = f'(\alpha)g(\alpha) + f(\alpha)g'(\alpha)$;

   Quotient Rule $\quad (f/g)'(\alpha) = \dfrac{g(\alpha)f'(\alpha) - f(\alpha)g'(\alpha)}{(g(\alpha))^2}$,
   provided that $g(\alpha) \neq 0$.

7. A polynomial function
   $$p(z) = a_0 + a_1 z + \cdots + a_n z^n \quad (z \in \mathbb{C}),$$
   where $a_0, a_1, \ldots, a_n \in \mathbb{C}$ and $a_n \neq 0$, is entire with derivative
   $$p'(z) = a_1 + 2a_2 z + \cdots + n a_n z^{n-1} \quad (z \in \mathbb{C}).$$
   Any rational function is analytic.

8. **Strategy A for non-differentiability**
   If f is discontinuous at α, then f is not differentiable at α.

9. **Strategy B for non-differentiability**
   To prove that a function f is not differentiable at α, apply the strategy for the non-existence of limits to the difference quotient
   $$\frac{f(z) - f(\alpha)}{z - \alpha}.$$

10. **Higher-order derivatives** of a function f are obtained by repeated differentiation:
    $$(f')' = f'' = f^{(2)},$$
    $$(f'')' = f''' = f^{(3)},$$
    and so on.
    The nth derivative of f is denoted by $f^{(n)}$.

11. **A geometric interpretation of derivatives**
    If $f'(\alpha) \neq 0$, then, to a close approximation,
    $$f(z) - f(\alpha) \simeq f'(\alpha)(z - \alpha).$$
    It follows that, to a close approximation, a small disc centred at α is mapped to a small disc centred at $f(\alpha)$. In this process, the disc is rotated through the angle $\operatorname{Arg} f'(\alpha)$, and scaled by the factor $|f'(\alpha)|$.

Section 2: The Cauchy–Riemann Equations

1. **Cauchy–Riemann Theorem**

   If f is differentiable at $\alpha = a + ib$, then $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}$,
   $\dfrac{\partial v}{\partial y}$ exist at (a, b) and satisfy the **Cauchy–Riemann equations**

   $$\frac{\partial u}{\partial x}(a, b) = \frac{\partial v}{\partial y}(a, b) \quad \text{and} \quad \frac{\partial v}{\partial x}(a, b) = -\frac{\partial u}{\partial y}(a, b).$$

2. **Strategy C for non-differentiability**

   If $f = u + iv$, then f is non-differentiable at $a + ib$. If either
   $$\frac{\partial u}{\partial x}(a, b) \neq \frac{\partial v}{\partial y}(a, b) \quad \text{or} \quad \frac{\partial v}{\partial x}(a, b) \neq -\frac{\partial u}{\partial y}(a, b),$$
   then f is not differentiable at $a + ib$.

3. **Cauchy–Riemann Converse Theorem**

   Let $f(x + iy) = u(x, y) + iv(x, y)$ be defined on a region $\mathcal{R}$ containing $a + ib$. If the partial derivatives
   $$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$$

   1. exist on $\mathcal{R}$,
   2. are continuous at (a, b),
   3. satisfy the Cauchy–Riemann equations at (a, b),

   then f is differentiable at $a + ib$ and
   $$f'(a + ib) = \frac{\partial u}{\partial x}(a, b) + i\frac{\partial v}{\partial x}(a, b).$$

Section 3: The Composition, Inverse and Restriction Rules

1. **Composition Rule**

   Let f and g be complex functions, and let α be a limit point of the domain of $g \circ f$. If f is differentiable at α, and g is differentiable at $f(\alpha)$, then $g \circ f$ is differentiable at α, and
   $$(g \circ f)'(\alpha) = g'(f(\alpha)) f'(\alpha).$$

2. **Inverse Function Rule**

   Let $f : A \longrightarrow B$ be a one–one complex function, and suppose that f^{-1} is continuous at $\beta \in B$. If f has a non-zero derivative at $f^{-1}(\beta) \in A$, then f^{-1} is differentiable at β and $\left(f^{-1}\right)'(\beta) = \dfrac{1}{f'(f^{-1}(\beta))}$.

3. **Restriction Rule**

   Let f and g be functions with domains A and B, respectively, and let $A \subseteq B$. If $\alpha \in A$ is a limit point of A and

   1. $f(z) = g(z)$, for $z \in A$,
   2. g is differentiable at α,

   then f is differentiable at α, and $f'(\alpha) = g'(\alpha)$.

4. **Standard derivatives**

Rule of f	Rule of f'	Domain of f'
$\alpha, \alpha \in \mathbf{C}$	0	$\mathbf{C}$
$z^k, k \in \mathbf{Z}, k > 0$	kz^{k-1}	$\mathbf{C}$
$z^k, k \in \mathbf{Z}, k < 0$	kz^{k-1}	$\mathbf{C} - \{0\}$
$z^k, k \in \mathbf{C} - \mathbf{Z}$	kz^{k-1}	$\mathbf{C} - \{x \in \mathbf{R} : x \leq 0\}$
$\exp z$	$\exp z$	$\mathbf{C}$
$\text{Log } z$	$1/z$	$\mathbf{C} - \{x \in \mathbf{R} : x \leq 0\}$
$\sin z$	$\cos z$	$\mathbf{C}$
$\cos z$	$-\sin z$	$\mathbf{C}$
$\tan z$	$\sec^2 z$	$\mathbf{C} - \left\{\left(n + \frac{1}{2}\right)\pi : n \in \mathbf{Z}\right\}$
$\sinh z$	$\cosh z$	$\mathbf{C}$
$\cosh z$	$\sinh z$	$\mathbf{C}$
$\tanh z$	$\text{sech}^2 z$	$\mathbf{C} - \left\{\left(n + \frac{1}{2}\right)\pi i : n \in \mathbf{Z}\right\}$

Section 4: Smooth Paths

1. Let Γ be a path with parametrization $\gamma : I \longrightarrow \mathbf{C}$, and suppose that $c \in I$. If γ is differentiable at c and if $\gamma'(c) \neq 0$, then $\gamma'(c)$ may be interpreted as a tangent vector to the path Γ at the point $\gamma(c)$.

2. Let ϕ and ψ be real functions, both with domain some interval I. Then the parametrization
   $$\gamma(t) = \phi(t) + i\psi(t) \quad (t \in I)$$
   is differentiable at a point $c \in I$ if and only if both ϕ and ψ are differentiable at c. If γ is differentiable at c, then
   $$\gamma'(c) = \phi'(c) + i\psi'(c).$$

3. A parametrization $\gamma : I \longrightarrow \mathbf{C}$ is **smooth** if

   (a) γ is differentiable on I;

   (b) γ' is continuous on I;

   (c) γ' is non-zero on I.
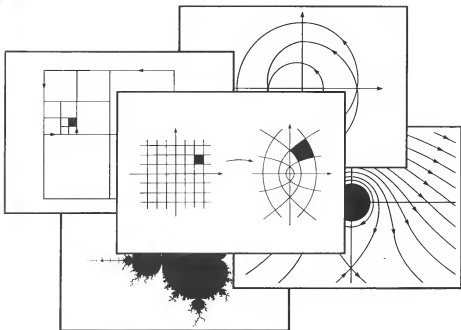
   A path is **smooth** if its parametrization is smooth.

4. Let f be a function that is analytic on a region $\mathcal{R}$, and suppose that $f'(\alpha) \neq 0$ for some $\alpha \in \mathcal{R}$. If Γ is a smooth path in $\mathcal{R}$, passing through α, then the tangent vector to $f(\Gamma)$ at $f(\alpha)$ may be obtained from the tangent vector to Γ at α by a rotation through the angle $\text{Arg } f'(\alpha)$ and a scaling by the factor $|f'(\alpha)|$.

5. A function is **conformal** at α if it leaves unchanged the size and orientation of the angle between any two smooth paths through α. A function is **conformal** on a set S if it is conformal at every point of S. A function is conformal if it is conformal throughout its domain.

6. Let f be a function which is analytic at α, with $f'(\alpha) \neq 0$. Then f is conformal at α.

7. Smooth paths that meet at right angles in $\mathbf{C}$ are said to be **orthogonal**. A grid made up of orthogonal smooth paths is called an **orthogonal grid**.

PART III: INDEX

The Open University

COMPLEX ANALYSIS

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1.1

CONTENTS

PART I: UNIT SUMMARIES

Unit B1 Integration

Section 1: Integrating Real Functions

This section revises the definition and main properties of the integration of real functions.

Section 2: Integrating Complex Functions

1. Let $\Gamma : \gamma(t)$ $(t \in [a, b])$ be a smooth path in $\mathbf{C}$, and let f be a function which is continuous on Γ. Then the integral of f along the path Γ, denoted by $\int_{\Gamma} f(z)\, dz$ or $\int_{\Gamma} f$, is

$$\int_{\Gamma} f(z)\, dz = \int_a^b f(\gamma(t)) \gamma'(t)\, dt.$$

This integral is evaluated as follows:

$$\int_a^b f(\gamma(t)) \gamma'(t)\, dt = \int_a^b u(t)\, dt + i \int_a^b v(t)\, dt,$$

where $u(t) = \operatorname{Re}(f(\gamma(t))\gamma'(t))$ and $v(t) = \operatorname{Im}(f(\gamma(t))\gamma'(t))$.

2. The above definition may be remembered by writing
$$z = \gamma(t), \quad dz = \gamma'(t)\, dt.$$

3. Two parametrizations
$$\gamma_1 : [a_1, b_1] \longrightarrow \mathbf{C} \quad \text{and} \quad \gamma_2 : [a_2, b_2] \longrightarrow \mathbf{C}$$
are equivalent if there is a function $h : [a_2, b_2] \longrightarrow [a_1, b_1]$ satisfying
   (a) $h(a_2) = a_1$, $h(b_2) = b_1$,
   (b) h' exists on $[a_2, b_2]$, and is continuous and positive there,
such that
$$\gamma_1(t) = \gamma_2(h(t)), \quad \text{for } t \in [a_2, b_2].$$
If γ_1 and γ_2 are equivalent parametrizations, then their paths $\Gamma : \gamma_1$ and $\Gamma : \gamma_2$ are equivalent.

4. If the smooth paths $\Gamma : \gamma_1$ and $\Gamma : \gamma_2$ are equivalent and the function f is continuous on Γ, then $\int_{\Gamma} f(z)\, dz$ does not depend on which parametrization, γ_1 or γ_2, is used.

5. A **contour** Γ is a path which can be subdivided into a finite number of smooth paths $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$, joined end to end; the order of these constituent smooth paths is indicated by writing
$$\Gamma = \Gamma_1 + \Gamma_2 + \cdots + \Gamma_n.$$
The initial point of Γ is the initial point of Γ_1, and the final point of Γ is the final point of Γ_n.

6. Let $\Gamma = \Gamma_1 + \Gamma_2 + \cdots + \Gamma_n$ be a contour, and let f be a function which is continuous on Γ. Then the **(contour) integral** of f along Γ, denoted by $\int_{\Gamma} f(z)\, dz$ or $\int_{\Gamma} f$, is

$$\int_{\Gamma} f(z)\, dz = \int_{\Gamma_1} f(z)\, dz + \int_{\Gamma_2} f(z)\, dz + \cdots$$
$$+ \int_{\Gamma_n} f(z)\, dz.$$

7. All splittings of a contour into smooth paths give the same value for the contour integral.

8. **Combination Rules**
Let Γ be a contour, and let f and g be functions which are continuous on Γ. Then

Sum Rule
$$\int_{\Gamma} (f(z) + g(z))\, dz = \int_{\Gamma} f(z)\, dz + \int_{\Gamma} g(z)\, dz;$$

Multiple Rule
$$\int_{\Gamma} \lambda f(z)\, dz = \lambda \int_{\Gamma} f(z)\, dz, \quad \text{for } \lambda \in \mathbf{C}.$$

9. Let $\gamma(t)$ $(t \in [a, b])$ be a smooth path. Then the reverse path, denoted by $\widetilde{\Gamma}$, is the path with parametrization $\widetilde{\gamma}(t)$, where
$$\widetilde{\gamma}(t) = \gamma(a + b - t) \quad (t \in [a, b]).$$
As sets, Γ and $\widetilde{\Gamma}$ are the same.

10. If $\Gamma = \Gamma_1 + \Gamma_2 + \cdots + \Gamma_n$ is a contour, then the reverse contour $\widetilde{\Gamma}$ of Γ is
$$\widetilde{\Gamma} = \widetilde{\Gamma}_n + \widetilde{\Gamma}_{n-1} + \cdots + \widetilde{\Gamma}_1.$$

11. **Reverse Contour Theorem**
Let Γ be a contour, and let f be a function which is continuous on Γ. Then, if $\widetilde{\Gamma}$ is the reverse contour of Γ,
$$\int_{\widetilde{\Gamma}} f(z)\, dz = -\int_{\Gamma} f(z)\, dz.$$

Section 3: Evaluating Contour Integrals

1. Let f and F be functions defined on a region $\mathcal{R}$. Then F is a **primitive** of f on $\mathcal{R}$ if F is analytic on $\mathcal{R}$ and
$$F'(z) = f(z), \quad \text{for all } z \in \mathcal{R}.$$
The function F is also called an **antiderivative** or **indefinite integral** of f on $\mathcal{R}$.

2. **Fundamental Theorem of Calculus**
Let the function f be continuous on $\mathcal{R}$, and let F be a primitive of f on $\mathcal{R}$, and let Γ be a contour in $\mathcal{R}$ with initial point α and final point β. Then
$$\int_{\Gamma} f(z)\, dz = F(\beta) - F(\alpha) = [F(z)]_\alpha^\beta.$$

3. The Chain Rule,
$$\frac{d}{dz}(f(z))f'(z) = (g \circ f)'(z),$$
provides a primitive $g \circ f$ for the integrand in
$$\int_{\Gamma} \frac{d}{dz}(f(z))f'(z)\,dz.$$
This is a form of integration by substitution.

4. **Contour Independence Theorem**

Let the function f be continuous and have a primitive on a region $\mathcal{R}$, and let Γ_1 and Γ_2 be contours in $\mathcal{R}$ with the same initial point α and the same final point β. Then
$$\int_{\Gamma_1} f(z)\,dz = \int_{\Gamma_2} f(z)\,dz.$$

5. **Integration by Parts**

Let the functions f and g be analytic on a region $\mathcal{R}$, and let f' and g' be continuous on $\mathcal{R}$. Let Γ be a contour in $\mathcal{R}$ with initial point α and final point β. Then
$$\int_{\Gamma} f(z)g'(z)\,dz = [f(z)g(z)]_{\alpha}^{\beta} - \int_{\Gamma} f'(z)g(z)\,dz.$$

6. The Fundamental Theorem of Calculus cannot be used to integrate non-differentiable functions.

7. A path or contour Γ is **closed** if its initial and final points coincide.

If Γ is a closed contour, then the value of any contour integral along Γ does not depend on the choice of initial (= final) point.

8. **Closed Contour Theorem**

Let the function f be continuous and have a primitive F on a region $\mathcal{R}$. Then
$$\int_{\Gamma} f(z)\,dz = 0,$$
for any closed contour Γ in $\mathcal{R}$.

9. A **grid path** is a contour each of whose constituent smooth paths is a line segment parallel to either the real axis or the imaginary axis.

10. **Grid Path Theorem**

If $\mathcal{R}$ is a region, then any two points in $\mathcal{R}$ can be joined by a grid path in $\mathcal{R}$.

11. **Zero Derivative Theorem**

Let the function F be analytic on a region $\mathcal{R}$, and let $F'(z) = 0$, for all z in $\mathcal{R}$. Then F is constant on $\mathcal{R}$.

Section 4: Estimating Contour Integrals

1. Let $\Gamma: \gamma(t)$ $(t \in [a, b])$ be a smooth path. Then the length of the path Γ, denoted by $L(\Gamma)$, is
$$L(\Gamma) = \int_a^b |\gamma'(t)|\,dt.$$
The length of a contour is the sum of the lengths of its constituent smooth paths.

2. Equivalent smooth paths have the same length.

A smooth path and its reverse path have the same length.

3. **Estimation Theorem**

Let f be a function which is continuous on a contour Γ of length L, with
$$|f(z)| \le M, \quad \text{for } z \in \Gamma.$$
Then
$$\left| \int_{\Gamma} f(z)\,dz \right| \le ML.$$

4. Let $g: [a, b] \longrightarrow \mathbb{C}$ be a continuous function. Then
$$\left| \int_a^b g(t)\,dt \right| \le \int_a^b |g(t)|\,dt.$$

Unit B2   Cauchy's Theorem

Section 1: Cauchy's Theorem

1. A path $\Gamma : \gamma(t)$ $(t \in [a, b])$ is simple-closed if it is closed and γ is one-one on $[a, b)$.

   A path $\Gamma : \gamma(t)$ $(t \in [a, b])$ is simple if γ is one-one on $[a, b]$.

2. **Jordan Curve Theorem**

   If Γ is a simple-closed path, then the complement $\mathbb{C} - \Gamma$ of Γ is the union of two disjoint regions:

   a bounded region, called the inside of Γ, and

   an unbounded region, called the outside of Γ.

3. A region $\mathcal{R}$ is simply-connected if, whenever Γ is a simple-closed path lying in $\mathcal{R}$, the inside of Γ also lies in $\mathcal{R}$.

   In practice, we usually employ the following *informal definition* when testing a region for simple-connectedness:

   a region is simply-connected if it has no holes in it.

4. **Cauchy's Theorem**

   Let $\mathcal{R}$ be a simply-connected region, and let f be a function which is analytic on $\mathcal{R}$. Then

   $$\int_{\Gamma} f(z)\,dz = 0,$$

   for any closed contour Γ in $\mathcal{R}$.

5. **Contour Independence Theorem**

   Let $\mathcal{R}$ be a simply-connected region, let f be a function which is analytic on $\mathcal{R}$, and let Γ_1 and Γ_2 be contours in $\mathcal{R}$ with the same initial point α and the same final point β. Then

   $$\int_{\Gamma_1} f(z)\,dz = \int_{\Gamma_2} f(z)\,dz.$$

6. **Convention**   Unless otherwise specified, any simple-closed contour in a contour integral will be assumed to be traversed once anticlockwise.

7. **Shrinking Contour Theorem**

   Let $\mathcal{R}$ be a simply-connected region, let Γ be a simple-closed contour in $\mathcal{R}$, let α be a point inside Γ, and let g be a function which is analytic on $\mathcal{R} - \{\alpha\}$. Then

   $$\int_{\Gamma} g(z)\,dz = \int_{C} g(z)\,dz,$$

   where C is any circle with centre α, lying inside Γ.

Section 2: The Integral Formula

1. **Cauchy's Integral Formula**

   Let $\mathcal{R}$ be a simply-connected region, let Γ be a simple-closed contour in $\mathcal{R}$, and let f be a function which is analytic on $\mathcal{R}$. Then

   $$f(\alpha) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{z - \alpha}\,dz,$$

   for any point α inside Γ.

2. **Liouville's Theorem**

   If f is a bounded entire function, then f is constant.

Section 3: The Derivative Formulas

1. **Cauchy's nth Derivative Formula**

   Let $\mathcal{R}$ be a simply-connected region, let Γ be a simple-closed contour in $\mathcal{R}$, and let f be a function which is analytic on $\mathcal{R}$. Then f is n-times differentiable at α and

   $$f^{(n)}(\alpha) = \frac{n!}{2\pi i} \int_{\Gamma} \frac{f(z)}{(z - \alpha)^{n+1}}\,dz, \quad \text{for } n = 1, 2, \ldots.$$

2. **Analyticity of Derivatives**

   Let $\mathcal{R}$ be a region, and let f be a function which is analytic on $\mathcal{R}$. Then f possesses derivatives of all orders on $\mathcal{R}$, so that $f', f'', f^{(3)}, \ldots$ are analytic on $\mathcal{R}$.

Section 4: Revision

1. Contour integrals may be evaluated by the following methods:

   parametrization (using the definition of a
   contour integral) — *Unit B1*,

   Closed Contour Theorem — *Unit B1*,

   Cauchy's Theorem — Section 1,

   Cauchy's Integral Formula — Section 2,

   Cauchy's Derivative Formula — Section 3.

2. If the value of a contour integral is known, then evaluating the integral using parametrization yields the value of two real integrals.

3. **Strategy** To evaluate $I = \oint_\Gamma \dfrac{g(z)}{p(z)}\,dz$, where g is analytic on a simply-connected region containing a simple-closed contour Γ, and p is a polynomial function with no zeros on Γ:

(a) factorize $p(z)$ as $q(z)r(z)$, where the zeros of q lie inside Γ and those of r lie outside Γ; then $f = g/r$ is analytic on $\mathcal{R}$, a simply-connected region which contains Γ;

(b) expand $1/q(z)$ in partial fractions;

(c) expand $I = \oint_\Gamma \dfrac{f(z)}{q(z)}\,dz$ as a sum of integrals, each of which can be evaluated using Cauchy's Integral Formula and/or Cauchy's nth Derivative Formula.

Section 5: The proof of Cauchy's Theorem

1. **Cauchy's Theorem for a rectangular contour**
Let $\mathcal{R}$ be a simply-connected region, and let f be a function which is analytic on $\mathcal{R}$. Then
$$\int_{\Gamma} f(z)\,dz = 0,$$
for any rectangular contour Γ in $\mathcal{R}$.
(This theorem is used in the proof of the following theorem.)

2. **Cauchy's Theorem for a closed grid path**
Let $\mathcal{R}$ be a simply-connected region, and let f be a function which is analytic on $\mathcal{R}$. Then
$$\int_{\Gamma} f(z)\,dz = 0,$$
for any closed grid path Γ in $\mathcal{R}$.
(This theorem is used in the proof of the following theorem.)

3. **Primitive Theorem (or Antiderivative Theorem)**
If a function f is analytic on a simply-connected region $\mathcal{R}$, then f has a primitive on $\mathcal{R}$.

4. **Cauchy's Theorem** (see Section 1) is proved by using the Primitive Theorem and then appealing to the Closed Contour Theorem (*Unit B1*).

5. **Morera's Theorem**
If a function f is continuous on a region $\mathcal{R}$ and $\int_{\Gamma} f(z)\,dz = 0$, for all rectangular contours Γ in $\mathcal{R}$, then f is analytic on $\mathcal{R}$.

Unit B3 Taylor Series

Section 1: Complex Series

1. The expression $z_1 + z_2 + z_3 + \cdots$ is called an **infinite series** or a **series**. The number z_n is called the nth **term** of the series. The nth **partial sum** of the series is
$$s_n = z_1 + z_2 + \cdots + z_n = \sum_{k=1}^{n} z_k.$$

2. The series $z_1 + z_2 + z_3 + \cdots$ is **convergent** with sum s if the sequence $\{s_n\}$ of partial sums converges to s. We say that the series **converges to** s, and write
$$z_1 + z_2 + z_3 + \cdots = s \quad \text{or} \quad \sum_{n=1}^{\infty} z_n = s.$$
The series **diverges** if the sequence $\{s_n\}$ diverges.

3. If $\displaystyle\sum_{n=1}^{\infty} z_n$ is a convergent series, then $\{z_n\}$ is a null sequence.

4. **Non-null Test**
If the sequence $\{z_n\}$ is not null, then the series
$$\sum_{n=1}^{\infty} z_n \text{ is divergent.}$$

5. The series $\displaystyle\sum_{n=0}^{\infty} az^n = a + az + az^2 + \cdots$, where $a \in \mathbb{C}$, is called a **geometric series** with common ratio z.

6. **Geometric series**

(a) If $|z| < 1$ and $a \in \mathbb{C}$, then the series $\displaystyle\sum_{n=0}^{\infty} az^n$ is convergent with sum $a/(1-z)$.

(b) If $|z| \geq 1$ and $a \in \mathbb{C} - \{0\}$, then the series
$$\sum_{n=0}^{\infty} az^n \text{ is divergent.}$$

7. The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$ is called the **harmonic series**.

8. The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$ converges if $p > 1$, and diverges if $p \leq 1$.

9. **Combination Rules**
If $\displaystyle\sum_{n=1}^{\infty} z_n = s$ and $\displaystyle\sum_{n=1}^{\infty} w_n = t$, then

Sum Rule $\displaystyle\sum_{n=1}^{\infty} (z_n + w_n) = s + t;$

Multiple Rule $\displaystyle\sum_{n=1}^{\infty} \lambda z_n = \lambda s, \quad \text{for } \lambda \in \mathbb{C}.$

10. The series $\sum_{n=1}^{\infty} z_n$ is convergent if and only if both

the series $\sum_{n=1}^{\infty} \mathrm{Re}\, z_n$ and $\sum_{n=1}^{\infty} \mathrm{Im}\, z_n$ are convergent. In

this case

$$\sum_{n=1}^{\infty} z_n = \sum_{n=1}^{\infty} \mathrm{Re}\, z_n + i \sum_{n=1}^{\infty} \mathrm{Im}\, z_n.$$

11. Comparison Test

If $\sum_{n=1}^{\infty} a_n$ is a real convergent series of positive terms,

and if

$$|z_n| \le a_n, \quad \text{for } n = 1, 2, \ldots,$$

then the series $\sum_{n=1}^{\infty} z_n$ is convergent.

12. The series $\sum_{n=1}^{\infty} z_n$ is absolutely convergent if the

real series $\sum_{n=1}^{\infty} |z_n|$ is convergent.

13. Absolute Convergence Test

If the series $\sum_{n=1}^{\infty} z_n$ is absolutely convergent, then the

series $\sum_{n=1}^{\infty} z_n$ is convergent.

14. Triangle Inequality

If the series $\sum_{n=1}^{\infty} z_n$ is absolutely convergent, then

$$\left| \sum_{n=1}^{\infty} z_n \right| \le \sum_{n=1}^{\infty} |z_n|.$$

15. Ratio Test

Suppose that $\sum_{n=1}^{\infty} z_n$ is a complex series for which

$$\left| \frac{z_{n+1}}{z_n} \right| \to l \text{ as } n \to \infty.$$

(a) If $0 \le l < 1$, then $\sum_{n=1}^{\infty} z_n$ is absolutely

convergent.

(b) If $l > 1$, then $\sum_{n=1}^{\infty} z_n$ is divergent.

The case $l > 1$ includes the situation where

$\left| \frac{z_{n+1}}{z_n} \right| \to \infty$ as $n \to \infty$.

Section 2: Power Series

1. Let $z \in \mathbb{C}$. An expression of the form

$$\sum_{n=0}^{\infty} a_n (z - \alpha)^n = a_0 + a_1 (z - \alpha) + a_2 (z - \alpha)^2$$
$$+ \cdots,$$

where $\alpha \in \mathbb{C}$, $a_n \in \mathbb{C}$, $n = 0, 1, 2, \ldots$, is called a
power series about α.

2. A power series converges on a set S if for each
$z \in S$, the corresponding series converges.

3. Let $A = \{z : \sum_{n=0}^{\infty} a_n (z - \alpha)^n \text{ converges}\}$. The function

$$f(z) = \sum_{n=0}^{\infty} a_n (z - \alpha)^n \quad (z \in A)$$

is called the sum function of the power series.

4. Radius of Convergence Theorem

For a given power series

$$\sum_{n=0}^{\infty} a_n (z - \alpha)^n = a_0 + a_1 (z - \alpha) + a_2 (z - \alpha)^2$$
$$+ \cdots,$$

precisely one of the following possibilities occurs:

(a) the series converges only for $z = \alpha$;

(b) the series converges for all z;

(c) there is a number $R > 0$ such that

$$\sum_{n=0}^{\infty} a_n (z - \alpha)^n \text{ converges (absolutely)}$$

if $|z - \alpha| < R$,

and

$$\sum_{n=0}^{\infty} a_n (z - \alpha)^n \text{ diverges if } |z - \alpha| > R.$$

5. The positive real number R in case (c) of the Radius
of Convergence Theorem is called the radius of
convergence of the power series. Case (a)
corresponds to writing $R = 0$, and case (b) to writing
$R = \infty$.

6. All the convergence tests of Section 1 can be applied
to power series. In particular, the Ratio Test can be
used to find the radius of convergence of the power

series $\sum_{n=0}^{\infty} a_n (z - \alpha)^n$ provided that

$$\left| \frac{a_{n+1} (z - \alpha)^{n+1}}{a_n (z - \alpha)^n} \right|$$

tends to a limit (possibly ∞) as $n \to \infty$. In this case,
the radius of convergence is

$$R = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right|.$$

7. Let R be the radius of convergence of the power series
$$\sum_{n=0}^{\infty} a_n (z - \alpha)^n.$$
The *disc of convergence* of the power series is the open disc $\{z : |z - \alpha| < R\}$. The disc of convergence is interpreted to be $\varnothing$ if $R = 0$, and to be $\mathbb{C}$ if $R = \infty$.

8. A power series may converge at none, some or all of the points on the boundary of its disc of convergence.

9. **Differentiation Rule**
The power series
$$\sum_{n=0}^{\infty} a_n (z - \alpha)^n \quad \text{and} \quad \sum_{n=1}^{\infty} n a_n (z - \alpha)^{n-1}$$
have the same radius of convergence R, say.
Furthermore, if $f(z) = \sum_{n=0}^{\infty} a_n (z - \alpha)^n$, then f is analytic on the disc of convergence $\{z : |z - \alpha| < R\}$, and
$$f'(z) = \sum_{n=1}^{\infty} n a_n (z - \alpha)^{n-1}, \quad \text{for } |z - \alpha| < R.$$

10. **Integration Rule**
The power series
$$\sum_{n=0}^{\infty} a_n (z - \alpha)^n \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{a_n}{n+1} (z - \alpha)^{n+1}$$
have the same radius of convergence R, say.
Furthermore, if $f(z) = \sum_{n=0}^{\infty} a_n (z - \alpha)^n$, then the function
$$F(z) = \text{constant} + \sum_{n=0}^{\infty} \frac{a_n}{n+1} (z - \alpha)^{n+1}$$
is a primitive of f on $\{z : |z - \alpha| < R\}$.

Section 3: Taylor's Theorem

1. **Taylor's Theorem**
If f is a function which is analytic on the open disc $D = \{z : |z - \alpha| < r\}$, then
$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\alpha)}{n!} (z - \alpha)^n, \quad \text{for } z \in D. \qquad (*)$$
Moreover, this representation of f is unique, in the sense that if
$$f(z) = \sum_{n=0}^{\infty} a_n (z - \alpha)^n, \quad \text{for } z \in D,$$
then $a_n = f^{(n)}(\alpha)/n!$, for $n = 0, 1, 2, \dots$.
(In $(*)$, we have employed the conventions that $0! = 1, 0^0 = 1$ and $f^{(0)}(z) = f(z)$.)

2. If f is a function with derivatives $f'(\alpha), f''(\alpha), f'''(\alpha), \dots$ at the point α, then the power series
$$\sum_{n=0}^{\infty} \frac{f^{(n)}(\alpha)}{n!} (z - \alpha)^n$$
is called the *Taylor series* about α *for* f. The coefficient $f^{(n)}(\alpha)/n!$ is known as the nth *Taylor coefficient* (of f at α).

3. If f is an entire function, then the Taylor series about any point α for f converges to $f(z)$ for all $z \in \mathbb{C}$.

4. A function $f : A \longrightarrow \mathbb{C}$ is *even* if
$$f(-z) = f(z), \quad \text{for } z \in A,$$
and *odd* if
$$f(-z) = -f(z), \quad \text{for } z \in A.$$
The Taylor series about 0 of an even (odd) function includes only even (odd) powers of z.

5. **Basic Taylor Series**
$(1 - z)^{-1} = 1 + z + z^2 + z^3 + \cdots, \quad \text{for } |z| < 1;$
$\exp z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots, \quad \text{for } z \in \mathbb{C};$
$\text{Log}(1 + z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots, \quad \text{for } |z| < 1;$
$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots, \quad \text{for } z \in \mathbb{C};$
$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots, \quad \text{for } z \in \mathbb{C};$
$\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \cdots, \quad \text{for } z \in \mathbb{C};$
$\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \cdots, \quad \text{for } z \in \mathbb{C}.$

6. **Binomial series**
If $\alpha \in \mathbb{C}$, then the binomial series about 0 for the function $f(z) = (1 + z)^\alpha$ is
$$(1 + z)^\alpha = 1 + \binom{\alpha}{1} z + \binom{\alpha}{2} z^2 + \binom{\alpha}{3} z^3 + \cdots,$$
$$\text{for } |z| < 1,$$
where $\binom{\alpha}{n} = \dfrac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-(n-1))}{n!}.$

Section 4: Manipulating Taylor Series

1. The restrictions of the functions tan and sin to the region $S = \{z : -\pi/2 < \text{Re } z < \pi/2\}$ have inverse functions denoted by $\tan^{-1}$ and $\sin^{-1}$. The domains of both of these inverse functions contain the open unit disc with centre zero, and their derivatives have the following rules:
$$(\tan^{-1})'(z) = \frac{1}{1 + z^2}; \quad (\sin^{-1})'(z) = \frac{1}{\sqrt{1 - z^2}}.$$

2. Product Rule

Let

$$f(z) = \sum_{n=0}^{\infty} a_n (z - \alpha)^n, \quad \text{for } |z - \alpha| < R,$$

$$g(z) = \sum_{n=0}^{\infty} b_n (z - \alpha)^n, \quad \text{for } |z - \alpha| < R'.$$

If $r = \min\{R, R'\}$, then

$$(fg)(z) = \sum_{n=0}^{\infty} c_n (z - \alpha)^n, \quad \text{for } |z - \alpha| < r,$$

where, for each n, the numbers $c_0, c_1, \ldots, c_n$ are the coefficients of $(z - \alpha)^0, (z - \alpha)^1, \ldots, (z - \alpha)^n$ in

$$\left(\sum_{k=0}^{m} a_k (z - \alpha)^k \right) \left(\sum_{k=0}^{m} b_k (z - \alpha)^k \right).$$

3. Composition Rule

Let

$$f(z) = \sum_{n=0}^{\infty} a_n (z - \alpha)^n, \quad \text{for } |z - \alpha| < R,$$

$$g(z) = \sum_{n=0}^{\infty} b_n (w - \beta)^n, \quad \text{for } |w - \beta| < R'.$$

If $\beta = f(\alpha)$ (so $b_0 = a_0$), then, for some $r > 0$,

$$g(f(z)) = \sum_{n=0}^{\infty} c_n (z - \alpha)^n, \quad \text{for } |z - \alpha| < r,$$

where, for each n, the numbers $c_0, c_1, \ldots, c_n$ are the coefficients of $(z - \alpha)^0, (z - \alpha)^1, \ldots, (z - \alpha)^n$ in

$$\sum_{k=0}^{m} b_k \left(\sum_{i=1}^{m} a_i (z - \alpha)^i \right)^k.$$

4.

Taylor series may be deduced from known ones (for example, the basic Taylor series about 0 listed in Section 3) by:

(a) using the Combination Rules (Section 1);

(b) using the Product Rule;

(c) using the Composition Rule;

(d) using the Differentiation Rule (Section 2);

(e) using the Integration Rule (Section 2);

(f) using the substitution $w = \lambda z^k$ (where $\lambda \neq 0$ and $k \in \{1, 2, 3, \ldots\}$), for which

$$|w| < R \Longleftrightarrow |z| < \sqrt[k]{R/|\lambda|};$$

(g) using the substitution $w = z + (\beta - \alpha)$, which changes a Taylor series about β to one about α and for which

$$|w - \beta| < R \Longleftrightarrow |z - \alpha| < R;$$

(h) using appropriate combinations of methods (a)–(g).

Section 5: The Uniqueness Theorem

1.

Let a function f be analytic at α. If

$$f(\alpha) = f'(\alpha) = f''(\alpha) = \ldots = f^{(k-1)}(\alpha) = 0,$$

but $f^{(k)}(\alpha) \neq 0$,

then f has a zero at α of (finite) order k.

A simple zero is a zero of order 1.

2.

A function f is analytic at a point α, and has a zero of order k at α, if and only if, for some $r > 0$,

$$f(z) = (z - \alpha)^k g(z), \quad \text{for } |z - \alpha| < r,$$

where g is analytic at α and $g(\alpha) \neq 0$.

3.

If a function f is analytic on a region $\mathcal{R}$ and not identically zero there, then any zeros of f is of finite order.

4.

A zero α of a function f is said to be isolated if some disc centred at α contains no other zeros of f.

5. Isolated zeros

A zero of finite order is isolated.

6.

If a function f is analytic on a region $\mathcal{R}$ and if S is a set of zeros of f which has a limit point in $\mathcal{R}$, then f is identically zero on $\mathcal{R}$ (that is, $f(z) = 0$, for all $z \in \mathcal{R}$).

7. Uniqueness Theorem

Let the functions f and g be analytic on a region $\mathcal{R}$ and suppose that f agrees with g throughout a set $S \subseteq \mathcal{R}$, where S has a limit point in $\mathcal{R}$. Then f agrees with g throughout $\mathcal{R}$.

Unit B4   Laurent Series

Section 1: Singularities

1. A function f has an (isolated) **singularity** at the point α if f is analytic on a punctured open disc $\{z : 0 < |z - \alpha| < r\}$, where $r > 0$, but not at α itself.

2. Let f be a function with domain A, and suppose that α is a limit point of A. The function f **tends to infinity** as z tends to α if,

for each sequence $\{z_n\}$ in $A - \{\alpha\}$ such that $z_n \to \alpha$,
$$f(z_n) \to \infty$$

(or, equivalently, if,

for each positive number M, there is a positive number δ such that
$$|f(z)| \geq M, \quad \text{for } z \in A \text{ and } 0 < |z - \alpha| < \delta).$$

We write $f(z) \to \infty$ as $z \to \alpha$.

3. Suppose that a function f has a singularity at α. Then

(a) f has a **removable singularity** at α if there is a function g, analytic on a disc $|z - \alpha| < r$, and a positive number r, such that
$$f(z) = g(z), \quad \text{for } 0 < |z - \alpha| < r;$$

(b) f has a **pole of order** k at α if there is a function g, analytic with $g(\alpha) \neq 0$, and a positive number r, such that
$$f(z) = \frac{g(z)}{(z - \alpha)^k}, \quad \text{for } 0 < |z - \alpha| < r;$$

(c) f has an **essential singularity** at α if the singularity at α is neither removable nor a pole.

In case (a), g is an analytic extension of f to $\{z : |z - \alpha| < r\}$.

4. If a function f has a singularity at α and
1. $\lim_{z \to \alpha} f(z)$ does not exist,
2. $f(z)$ does not tend to infinity as $z \to \alpha$,
then α is an essential singularity of f.

Section 2: Laurent's Theorem

1. Let $z \in \mathbb{C}$. An expression of the form
$$\sum_{n = -\infty}^{\infty} a_n (z - \alpha)^n = \cdots + \frac{a_{-2}}{(z - \alpha)^2} + \frac{a_{-1}}{(z - \alpha)}$$
$$+ a_0 + a_1 (z - \alpha) + a_2 (z - \alpha)^2 + \cdots, \quad (*)$$

where $a_n \in \mathbb{C}$, for $n \in \mathbb{Z}$, is called an **extended power series** about α.

2. For a given z, the extended power series $(*)$ is convergent if $\sum_{n=0}^{\infty} a_n (z - \alpha)^n$, the **analytic part**, and $\sum_{n=1}^{\infty} a_{-n} (z - \alpha)^{-n}$, the **singular part**, are both convergent; its sum is found by adding the sums of these two series.

3. Let $A = \{z : \sum_{n=-\infty}^{\infty} a_n (z - \alpha)^n \text{ converges}\}$. The function
$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - \alpha)^n \quad (z \in A)$$
is called the **sum function** of the extended power series.

4. If the analytic part of an extended power series has disc of convergence $\{z : |z - \alpha| < r\}$ and the singular part converges on $\{z : |z - \alpha| > 1/r'\}$, then the extended power series has **annulus of convergence**
$$A = \{z : |z - \alpha| < r\} \cap \{z : |z - \alpha| > 1/r'\}.$$

Depending on the values of r and r', A may be an open annulus, a punctured disc, a punctured plane, the 'outside' of a closed disc, or the empty set.

5. Laurent's Theorem
If f is a function which is analytic on the open annulus
$$A = \{z : r_1 < |z - \alpha| < r_2\},$$
where $0 \leq r_1 < r_2 \leq \infty$, then
$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - \alpha)^n, \quad \text{for } z \in A, \quad (*)$$
where
$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - \alpha)^{n+1}} \, dz, \quad \text{for } n \in \mathbb{Z},$$

C being any circle with centre α lying in A.

Moreover, the representation $(*)$ of f is unique in A.

6. The representation $(*)$ is the **Laurent series** (about α) for the function f on A.

If A is a punctured open disc, then the representation $(*)$ is called the **Laurent series** about α for f.

7. A function f may have different Laurent series about α on different annuli.

8. If a function f has a singularity at α and if its Laurent series about α is
$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - \alpha)^n,$$
then

(a) f has a removable singularity at α if and only if $a_n = 0$, for all $n < 0$;

(b) f has a pole of order k at α if and only if $a_n = 0$, for all $n < -k$, and $a_{-k} \neq 0$;

(c) f has an essential singularity at α if and only if $a_n \neq 0$, for infinitely many $n < 0$.

9. A Laurent series may be obtained from known Taylor series (for example, by using substitution) and by using partial fractions.

Section 3: Behaviour near a Singularity

1. Let a function f have a singularity at α. Then the following statements are equivalent:

(A) f has a removable singularity at α;

(B) $\lim_{z \to \alpha} f(z)$ exists;

(C) f is bounded on $\{z : 0 < |z - \alpha| < r\}$, for some $r > 0$;

(D) $\lim_{z \to \alpha} (z - \alpha) f(z) = 0$.

2. Let a function f have a singularity at α. Then the following statements are equivalent:

(A) f has a pole of order k;

(B) $\lim_{z \to \alpha} (z - \alpha)^k f(z)$ exists, and is non-zero;

(C) $1/f$ has a removable singularity at α which, when removed, gives rise to a zero of order k at α.

3. Let a function f have a singularity at α. Then f has a pole at α if and only if
$$f(z) \to \infty \text{ as } z \to \alpha.$$

4. **Casorati–Weierstrass Theorem**

Suppose that a function f has an essential singularity at α. Let w be any complex number, and let ε and δ be positive real numbers. Then there exists $z \in \mathbb{C}$ such that
$$0 < |z - \alpha| < \delta \quad \text{and} \quad |f(z) - w| < \varepsilon.$$

Section 4: Evaluating Residues using Laurent Series

1. If a function f is analytic on $D = \{z : 0 < |z - \alpha| < r\}$, then, by Laurent's Theorem,
$$\int_C \frac{f(w)}{(w - \alpha)^{n+1}} \, dw = 2\pi i a_n,$$
where $\sum_{n=-\infty}^{\infty} a_n (z - \alpha)^n$ is the Laurent series about α for f, and C is any circle with centre α lying in D.

2. If a function f is analytic on a punctured disc D with centre α, then the coefficient a_{-1} in the Laurent series about α for f is called the residue of f at α, and is denoted by $\operatorname{Res}(f, \alpha)$. Hence
$$\int_C f(z) \, dz = 2\pi i \operatorname{Res}(f, \alpha),$$
where C is any circle with centre α lying in D.

PART II: INDEX

The Open University

COMPLEX ANALYSIS

HANDBOOK THREE

1.1

CONTENTS

PART I: UNIT SUMMARIES
Unit C1   Residues

Section 1: Calculating Residues

1. Suppose that an analytic function f has a singularity at the point α. Then

$$\text{Res}(f, \alpha) = \lim_{z \to \alpha} (z - \alpha) f(z),$$

provided that this limit exists.

2. g/h **Rule**

Let $f(z) = g(z)/h(z)$, where the functions g and h are analytic at the point α, $h(\alpha) = 0$, and $h'(\alpha) \neq 0$. Then

$$\text{Res}(f, \alpha) = g(\alpha)/h'(\alpha).$$

3. **Cover-up Rule**

Let $f(z) = \dfrac{p(z)}{(z - \alpha) q(z)}$, where the functions p and q are analytic at the point α, and $q(\alpha) \neq 0$. Then

$$\text{Res}(f, \alpha) = p(\alpha)/q(\alpha).$$

4. If a function f has a pole of order k at the point α, then the residue of f at α is given by

$$\text{Res}(f, \alpha) = \frac{1}{(k-1)!} \lim_{z \to \alpha} \left(\frac{d^{k-1}}{dz^{k-1}} \left((z - \alpha)^k f(z) \right) \right).$$

Section 2: The Residue Theorem

1. **The Residue Theorem**

Let $\mathcal{R}$ be a simply-connected region, and let f be a function which is analytic on $\mathcal{R}$ except for a finite number of singularities. Let Γ be a simple-closed contour in $\mathcal{R}$, not passing through any of these singularities. Then

$$\int_{\Gamma} f(z) \, dz = 2\pi i S,$$

where S is the sum of the residues of f at those singularities that lie inside Γ.

2. **Strategy for evaluating** $\displaystyle\int_0^{2\pi} \Phi(\cos t, \sin t) \, dt$

(a) Replace

$$\cos t \text{ by } \tfrac{1}{2}(z + z^{-1}), \sin t \text{ by } \tfrac{1}{2i}(z - z^{-1})$$

and dt by $\dfrac{1}{iz} dz$,

to obtain a contour integral of the form $\int_C f(z) \, dz$ around the unit circle $C = \{z : |z| = 1\}$.

(b) Locate the singularities of the function f lying inside C, and calculate the residues of f at these points.

(c) Evaluate the given integral by calculating

$$2\pi i \times (\text{the sum of the residues found in step (b)}).$$

Section 3: Evaluating Improper Integrals

1. Let f be a function defined on an unbounded interval $]a, \infty[$. Then the function f has limit α as x tends to ∞ if

for each sequence $\{r_n\}$ in $]a, \infty[$ such that $r_n \to \infty$,

$$f(r_n) \to \alpha \text{ as } n \to \infty.$$

In this case we write

EITHER $\displaystyle\lim_{r \to \infty} f(r) = \alpha$,

OR $\quad f(r) \to \alpha$ as $r \to \infty$.

2. **Combination Rules**

Let f and g be functions such that

$$\lim_{r \to \infty} f(r) = \alpha \quad \text{and} \quad \lim_{r \to \infty} g(r) = \beta.$$

Then

Sum Rule $\displaystyle\lim_{r \to \infty} (f(r) + g(r)) = \alpha + \beta$;

Multiple Rule $\displaystyle\lim_{r \to \infty} (\lambda f(r)) = \lambda \alpha$, for $\lambda \in \mathbb{C}$;

Product Rule $\displaystyle\lim_{r \to \infty} (f(r) g(r)) = \alpha \beta$;

Quotient Rule $\displaystyle\lim_{r \to \infty} (f(r)/g(r)) = \alpha/\beta$,

provided that $\beta \neq 0$.

3. If p and q are polynomial functions such that the degree of q exceeds the degree of p, then

$$\lim_{r \to \infty} \frac{p(r)}{q(r)} = 0.$$

4. Let f be a continuous function with domain $\mathbb{R}$. Then the improper integral $\displaystyle\int_{-\infty}^{\infty} f(t) \, dt$ is

$$\int_{-\infty}^{\infty} f(t) \, dt = \lim_{r \to \infty} \int_{-r}^{r} f(t) \, dt,$$

provided that this limit exists.

Let f be a function which is continuous on the interval $[a, \infty[$. Then the improper integral $\displaystyle\int_a^{\infty} f(t) \, dt$ is

$$\int_a^{\infty} f(t) \, dt = \lim_{r \to \infty} \int_a^{r} f(t) \, dt,$$

provided that this limit exists.

5. Let f be a continuous function with domain $\mathbb{R}$. Then:

(a) if f is an odd function, $\displaystyle\int_{-\infty}^{\infty} f(t) \, dt = 0$;

(b) if f is an even function,

$$\int_{-\infty}^{\infty} f(t) \, dt = 2 \int_0^{\infty} f(t) \, dt,$$

provided that these integrals exist.

6. Let a function f be defined and continuous at all points of an interval $[a, b]$ except at the point $c \in]a, b[$. Then the improper integral $\int_a^b f(t)\, dt$ is

$$\int_a^b f(t)\, dt = \lim_{\varepsilon \to 0} \left(\int_a^{c-\varepsilon} f(t)\, dt + \int_{c+\varepsilon}^b f(t)\, dt \right),$$

where the limit is taken as ε tends to 0 through positive values, provided that this limit exists.

Let a function f be continuous at all points of $\mathbb{R}$, except at the point c. Then the improper integral

$$\int_{-\infty}^{\infty} f(t)\, dt$$

is

$$\int_{-\infty}^{\infty} f(t)\, dt = \lim_{r \to \infty} \int_{-r}^r f(t)\, dt$$

$$= \lim_{r \to \infty} \left(\lim_{\varepsilon \to 0} \left(\int_{-r}^{c-\varepsilon} f(t)\, dt + \int_{c+\varepsilon}^r f(t)\, dt \right) \right),$$

provided that these limits exist.

7. Round-the-Pole Lemma

If a function f has a simple pole at α and Γ is the semicircular contour (traversed anticlockwise) from $\alpha + \varepsilon$ to $\alpha - \varepsilon$, then

$$\lim_{\varepsilon \to 0} \int_{\Gamma} f(z)\, dz = \pi i \operatorname{Res}(f, \alpha).$$

8. Let p and q be polynomial functions such that:

1. the degree of q exceeds that of p by at least two;
2. any poles of p/q on the real axis are simple.

Then

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)}\, dx = 2\pi i S + \pi i T,$$

where S is the sum of the residues of the function p/q at those poles in the upper half-plane, and T is the sum of residues of the function p/q at those poles on the real axis.

9. Let p and q be polynomial functions such that:

1. the degree of q exceeds that of p by at least one;
2. any poles of p/q on the real axis are simple.

Then, if $k > 0$,

$$\int_{-\infty}^{\infty} \frac{p(t)}{q(t)} e^{ikt}\, dt = 2\pi i S + \pi i T, \qquad (*)$$

where S is the sum of the residues of the function $z \longmapsto (p/q)(z)e^{ikz}$ at those poles in the upper half-plane, and T is the sum of the residues of the function $z \longmapsto (p/q)(z)e^{ikz}$ at those poles on the real axis.

By equating the real and imaginary parts of Equation $(*)$, we obtain the values of the real improper integrals

$$\int_{-\infty}^{\infty} \frac{p(t)}{q(t)} \cos kt\, dt \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{p(t)}{q(t)} \sin kt\, dt.$$

10. Jordan's Lemma

Let Γ be the semicircular contour (traversed anticlockwise) from r to $-r$ and suppose that a function f is continuous on Γ and satisfies

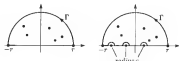$$|f(z)| \le M, \quad \text{for } z \in \Gamma.$$

Then, for $k > 0$, we have

$$\left| \int_{\Gamma} f(z) e^{ikz}\, dz \right| \le \frac{M\pi}{k}.$$

11. Improper integrals of the forms

$$\int_{-\infty}^{\infty} \frac{p(t)}{q(t)}\, dt, \quad \int_{-\infty}^{\infty} \frac{p(t)}{q(t)} \cos kt\, dt \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{p(t)}{q(t)} \sin kt\, dt,$$

which are the subject of the results in items 8 and 9, may also be evaluated by means of a five-step method:

(a) Consider an appropriate contour integral I, where the contour Γ is of one of the following forms.

(b) Evaluate I by using the Residue Theorem.

(c) Split up I into line segments and semicircles.

(d) Estimate the integral along the semicircle of radius r.

(e) Let $r \to \infty$, or $\varepsilon \to 0$ (if appropriate) in the equation resulting from step (c), and then deduce the value of the improper integral.

Section 4: Summing Series

1. Let ϕ be an even function which is analytic on $\mathbb{C}$ except for poles at the points $\alpha_1, \alpha_2, \ldots, \alpha_k$ (none of which is an integer), and possibly at 0, and let S_N be the square contour with vertices at $(N + \frac{1}{2})(\pm 1 \pm i)$. Suppose also that the function $f(z) = \pi \cot \pi z \cdot \phi(z)$ is such that

$$\lim_{N \to \infty} \int_{S_N} f(z)\, dz = 0.$$

Then

$$\sum_{n=1}^{\infty} \phi(n) = -\frac{1}{2} \left(\operatorname{Res}(f, 0) + \sum_{j=1}^k \operatorname{Res}(f, \alpha_j) \right).$$

Further, if ϕ is analytic at 0, then

$$\operatorname{Res}(f, 0) = \phi(0).$$

2. For each $N = 1, 2, \ldots,$

$$|\cot \pi z| \le 2, \quad \text{for } z \in S_N,$$

where S_N is the square contour with vertices at $(N + \frac{1}{2})(\pm 1 \pm i)$.

4

3. Let ϕ be an even function which is analytic on $\mathbb{C}$ except for poles at the points $\alpha_1, \alpha_2, \ldots, \alpha_k$ (none of which is an integer), and possibly at 0. Let S_N be the square contour with vertices at $(N + \frac{1}{2})(\pm 1 \pm i)$.

Suppose also that the function
$$f(z) = \pi \operatorname{cosec} \pi z \cdot \phi(z) \text{ is such that}$$
$$\lim_{N \to \infty} \int_{S_N} f(z)\, dz = 0.$$

Then
$$\sum_{n=1}^{\infty} (-1)^n \phi(n) = -\frac{1}{2}\left(\operatorname{Res}(f, 0) + \sum_{j=1}^{k} \operatorname{Res}(f, \alpha_j) \right).$$

Further, if ϕ is analytic at 0, then
$$\operatorname{Res}(f, 0) = \phi(0).$$

4. For each $N = 1, 2, \ldots,$
$$|\operatorname{cosec} \pi z| \le 1, \quad \text{for } z \in S_N,$$
where S_N is the square contour with vertices at $(N + \frac{1}{2})(\pm 1 \pm i)$.

5. The Laurent series about 0 for cot and cosec are:
$$\cot z = \frac{1}{z} - \frac{1}{3}z - \frac{1}{45}z^3 - \cdots;$$
$$\operatorname{cosec} z = \frac{1}{z} + \frac{1}{6}z + \frac{7}{360}z^3 + \cdots.$$

Unit C2   Zeros and Extrema

Section 1: The Winding Number

1. A **continuous argument function** for a path $\Gamma : \gamma(t)$ ($t \in [a, b]$) lying in $\mathbb{C} - \{0\}$ is a continuous function
$$\theta : [a, b] \longmapsto \mathbb{R}$$
such that for each $t \in [a, b]$, $\theta(t)$ is an argument of $\gamma(t)$.

Such a function θ satisfies
$$\frac{\gamma(t)}{|\gamma(t)|} = e^{i\theta(t)}, \quad \text{for } t \in [a, b].$$

2. Any path $\Gamma : \gamma(t)$ ($t \in [a, b]$) lying in $\mathbb{C} - \{0\}$ has a continuous argument function θ, which is unique apart from the addition of a constant term of the form $2\pi n$, where $n \in \mathbb{Z}$.

3. The **winding number** of a path $\Gamma : \gamma(t)$ ($t \in [a, b]$) lying in $\mathbb{C} - \{0\}$ round 0 is
$$\operatorname{Wnd}(\Gamma, 0) = \frac{1}{2\pi}(\theta(b) - \theta(a)),$$
where θ is any continuous argument function for Γ. $\operatorname{Wnd}(\Gamma, 0)$ may be calculated by inspecting a sketch of Γ, if one is available.

4. For $\phi \in \mathbb{R}$, the function Arg_ϕ is defined by
$$\operatorname{Arg}_\phi(z) = \tilde{\theta} \quad (z \in \mathbb{C} - \{0\}),$$
where $\tilde{\theta}$ is the argument of z lying in the interval $]\phi - 2\pi, \phi].$

5. For all $\phi \in \mathbb{R}$, Arg_ϕ is continuous on the cut plane
$$\mathbb{C}_\phi = \{ re^{i\phi} : r > 0, \phi - 2\pi < \theta < \phi \}.$$

6. For $\phi \in \mathbb{R}$, the function Log_ϕ is defined by
$$\operatorname{Log}_\phi z = \log_e |z| + i \operatorname{Arg}_\phi z \quad (z \in \mathbb{C} - \{0\}).$$

7. For all $\phi \in \mathbb{R}$, Log_ϕ is analytic on $\mathbb{C}_\phi$ with derivative Log'_ϕ given by
$$\operatorname{Log}'_\phi z = 1/z, \quad \text{for } z \in \mathbb{C}_\phi.$$

8. Let $\Gamma : \gamma(t)$ ($t \in [a, b]$) be a closed contour lying in $\mathbb{C} - \{0\}$. Then
$$\operatorname{Wnd}(\Gamma, 0) = \frac{1}{2\pi i} \int_\Gamma \frac{1}{z}\, dz.$$

9. The **winding number** of the path $\Gamma : \gamma(t)$ ($t \in [a, b]$) lying in $\mathbb{C} - \{\alpha\}$ round α is
$$\operatorname{Wnd}(\Gamma, \alpha) = \frac{1}{2\pi}(\theta_\alpha(b) - \theta_\alpha(a)),$$
where θ_α is a continuous argument function for Γ relative to α (that is, θ_α is continuous on $[a, b]$ and $\theta_\alpha(t)$ is an argument of $\gamma(t) - \alpha$).

10. $\mathrm{Wnd}(\Gamma, \alpha) = \mathrm{Wnd}(\Gamma - \alpha, 0)$,

where

$$\Gamma - \alpha : \gamma(t) - \alpha \quad (t \in [a, b])$$

is the path Γ translated by $-\alpha$.

11. Let Γ be a closed path and let D be an open set lying in the complement of Γ. Then the function $\alpha \longmapsto \mathrm{Wnd}(\Gamma, \alpha)$ is constant on D.

Section 2: Locating Zeros of Analytic Functions

1. Let an analytic function f have a zero of order n at α. Then the function f'/f has a simple pole at α with

$$\mathrm{Res}(f'/f, \alpha) = n.$$

2. Argument Principle

Let a function f be analytic on a simply-connected region $\mathcal{R}$ and let Γ be a simply-closed contour in $\mathcal{R}$, such that $f(z) \neq 0$, for $z \in \Gamma$. Then

$$\mathrm{Wnd}(f(\Gamma), 0) = N,$$

where N is the number of zeros of f inside Γ, counted according to their orders.

3. Let a function f be analytic on a simply-connected region $\mathcal{R}$ and let Γ be a simply-closed contour in $\mathcal{R}$, such that $f(z) \neq 0$ on Γ. Then $\mathrm{Wnd}(\Gamma, \beta)$ is the number of zeros of the function $f - \beta$ inside Γ, counted according to their orders.

Unless otherwise stated, zeros are always counted according to their orders.

4. Rouché's Theorem

Suppose that

1. the function f is analytic on a simply-connected region $\mathcal{R}$;

2. Γ is a simple-closed contour in $\mathcal{R}$ and

$$|f(z) - g(z)| < |g(z)|, \quad \text{for } z \in \Gamma.$$

Then f has the same number of zeros as g inside Γ. (The function g is known as a 'dominant term' in f on Γ.)

5. Fundamental Theorem of Algebra

Let $p(z) = a_0 + a_1 z + \cdots + a_n z^n$, where $n \geq 1$ and $a_n \neq 0$. Then p has exactly n zeros, all lying in the open disc $\{z : |z| < R\}$, where

$$R = 1 + \max\{|a_0|/|a_n|, \ldots, |a_{n-1}|/|a_n|\}.$$

Section 3: Local Behaviour of Analytic Functions

1. Open Mapping Theorem

Let the function f be analytic and non-constant on a region $\mathcal{R}$ and let G be an open subset of $\mathcal{R}$. Then $f(G)$ is open.

2. If a function f is analytic and non-constant on a region $\mathcal{R}$, then $f(\mathcal{R})$ is a region.

3. Let a function f be analytic on a region $\mathcal{R}$ and let $\alpha \in \mathcal{R}$. Then f is n-one near α if there is a region $\mathcal{S}$ in $\mathcal{R}$, with $\alpha \in \mathcal{S}$, and a function ϕ which is analytic and one-one on $\mathcal{S}$, such that

$$f(z) = f(\alpha) + (\phi(z))^n, \quad \text{for } z \in \mathcal{S}.$$

4. Local Mapping Theorem

Let a function f be analytic on a region $\mathcal{R}$ and let $\alpha \in \mathcal{R}$. Then the Taylor series about α has the form

$$f(z) = f(\alpha) + a_n(z - \alpha)^n + a_{n+1}(z - \alpha)^{n+1} + \cdots, \quad (*)$$

where $n \geq 1$ and $a_n \neq 0$, if and only if f is n-one near α.

5. The Taylor series for f about α takes the form $(*)$ if and only if

$$0 = f'(\alpha) = f''(\alpha) = \cdots = f^{(n-1)}(\alpha),$$

but $f^{(n)}(\alpha) \neq 0$.

6. Inverse Function Rule

Let f be a one-one analytic function whose domain is a region $\mathcal{R}$. Then f^{-1} is analytic on $f(\mathcal{R})$ and

$$\left(f^{-1}\right)'(\beta) = \frac{1}{f'(f^{-1}(\beta))}, \quad \text{for } \beta \in f(\mathcal{R}).$$

7. The inverse functions $\tan^{-1}$ and $\sin^{-1}$ are analytic.

8. Strategy for inverting a Taylor series

Given the Taylor series about α for f:

$$f(z) = \sum_{n=0}^{\infty} a_n(z - \alpha)^n,$$

where $a_1 = f'(\alpha) \neq 0$, we can find the Taylor series about $\beta = f(\alpha)$ for f^{-1}:

$$f^{-1}(w) = \sum_{n=0}^{\infty} b_n(w - \beta)^n,$$

by putting $b_0 = \alpha$ and equating the powers of $(z - \alpha)$ in the identity

$$z - \alpha = b_1(a_1(z - \alpha) + a_2(z - \alpha)^2 + \cdots) + b_2(a_1(z - \alpha) + a_2(z - \alpha)^2 + \cdots)^2 + \cdots,$$

to obtain equations for $b_1, b_2, \ldots$, in terms of $a_1, a_2, \ldots$.

Section 4: Extreme Values of Analytic Functions

1. Let a function f be defined on a region $\mathcal{R}$. Then the function $|f|$ has a local maximum at the point $\alpha \in \mathcal{R}$ if there is some $r \geq 0$ such that $\{z : |z - \alpha| < r\} \subseteq \mathcal{R}$ and

   $$|f(z)| \leq |f(\alpha)|, \quad \text{for } |z - \alpha| < r.$$

2. **Local Maximum Principle**

   Let a function f be analytic on a region $\mathcal{R}$. If f is non-constant on $\mathcal{R}$, then the function $|f|$ has no local maxima on $\mathcal{R}$.

3. The closure $\overline{A}$ of a set A in $\mathbb{C}$ is

   $$\overline{A} = \text{int } A \cup \partial A.$$

4. **Maximum Principle**

   Let a function f be analytic on a bounded region $\mathcal{R}$, and continuous on $\overline{\mathcal{R}}$. Then there exists $\alpha \in \partial \mathcal{R}$ such that

   $$|f(z)| \leq |f(\alpha)|, \quad \text{for } z \in \overline{\mathcal{R}}.$$

5. **Minimum Principle**

   Let a function f be analytic on a bounded region $\mathcal{R}$, and continuous and non-zero on $\overline{\mathcal{R}}$. Then there exists $\alpha \in \partial \mathcal{R}$ such that

   $$|f(z)| \geq |f(\alpha)|, \quad \text{for } z \in \overline{\mathcal{R}}.$$

6. **Boundary Uniqueness Theorem**

   Let functions f and g be analytic on a bounded region $\mathcal{R}$ and continuous on $\overline{\mathcal{R}}$. If $f = g$ on $\partial \mathcal{R}$, then $f = g$ on $\mathcal{R}$.

7. **Schwarz's Lemma**

   Let a function f be analytic on $\{z : |z| < R\}$ and $f(0) = 0$, and suppose that

   $$|f(z)| \leq M, \quad \text{for } |z| < R.$$

   Then

   $$|f(z)| \leq (M/R)|z|, \quad \text{for } |z| < R.$$

Unit C3 Analytic Continuation

Section 1: What is Analytic Continuation?

1. Let f and g be analytic functions whose domains are the regions $\mathcal{R}$ and $\mathcal{S}$, respectively. Then f and g are **direct analytic continuations** of each other if there is a region $\mathcal{T} \subseteq \mathcal{R} \cap \mathcal{S}$ such that

   $$f(z) = g(z), \quad \text{for } z \in \mathcal{T}.$$

   We also say that g is a direct analytic continuation of f from $\mathcal{R}$ to $\mathcal{S}$, and vice versa.

2. Let a function f be continuous on the interval $]0, \infty[$. Then

   $$\int_0^\infty f(t)\,dt = \lim_{\varepsilon \to 0} \int_\varepsilon^1 f(t)\,dt + \lim_{r \to \infty} \int_1^r f(t)\,dt,$$

   provided that both these limits exist.

3. **Improper integrals of the forms**

   $$\int_0^\infty \frac{p(z)}{q(z)} \log_e t\,dt, \int_0^\infty \frac{p(z)}{q(z)} t^a\,dt, \int_0^\infty \frac{p(z)}{q(z)}\,dz,$$

   where $0 < a < 1$ and p, q are polynomial functions such that p/q is even, the degree of q exceeds that of p by at least two and any poles of p/q on the non-negative real axis are simple, may be evaluated by a five-step method like that given in item 11 of *Unit C1*, Section 3 above.

4. Let p and q be polynomial functions such that

   1. the degree of q exceeds the degree of p by at least 2;
   2. any poles of p/q on the non-negative real axis are simple.

   Then, for $0 < a < 1$,

   $$\int_0^\infty \frac{p(z)}{q(z)} t^a\,dt = -(\pi e^{-\pi a i} \operatorname{cosec} \pi a) S - (\pi \cot \pi a) T,$$

   where S is the sum of the residues of the function

   $$f_1(z) = \frac{p(z)}{q(z)} \exp\left(a \operatorname{Log}_{2\pi}(z)\right)$$

   in $\mathbb{C}_{2\pi}$, and T is the sum of the residues of the function

   $$f_2(z) = \frac{p(z)}{q(z)} \exp\left(a \log z\right)$$

   on the positive real axis.

Section 2: Indirect Analytic Continuation

1. Let an analytic function f have as domain a region $\mathcal{R}$ and let $\alpha \in \mathcal{R}$. If the disc of convergence D of the Taylor series about α for f contains points which are not in $\mathcal{R}$, then the function g, with domain D, defined by the Taylor series is a **direct analytic continuation** of f by Taylor series.

2. For convenience an analytic function f whose domain is a region $\mathcal{R}$ is sometimes written $(f, \mathcal{R})$.

3. The finite sequence of functions
$$(f_1, \mathcal{R}_1), (f_2, \mathcal{R}_2), \ldots, (f_n, \mathcal{R}_n)$$
forms a **chain** if, for $k = 1, 2, \ldots, n-1$,
$$(f_{k+1}, \mathcal{R}_{k+1}) \text{ is a direct analytic continuation}$$
of $(f_k, \mathcal{R}_k)$.

Any two functions of the chain are called **analytic continuations** of each other, and the chain is said to **join** $(f_1, \mathcal{R}_1)$ to $(f_n, \mathcal{R}_n)$. If $\mathcal{R}_1 = \mathcal{R}_n$, then the chain is said to be **closed**.

Two functions of a chain which are not direct analytic continuations of each other are called **indirect analytic continuations**.

Section 3: Uniform Convergence

1. A sequence of functions $\{f_n\}$ **converges pointwise** (to a limit function f) on a set E if, for each $z \in E$,
$$\lim_{n \to \infty} f_n(z) = f(z).$$

2. A sequence of functions $\{f_n\}$ **converges uniformly** (to a limit function f) on a set E if
for each $\varepsilon > 0$, there is an integer N such that
$$|f_n(z) - f(z)| < \varepsilon, \text{ for all } n > N, \text{ and all } z \in E.$$
We also say that $\{f_n\}$ is uniformly convergent on E, with limit function f.

3. **Strategy for proving uniform convergence**
To prove that a sequence of functions $\{f_n\}$ converges uniformly on a set E:
(a) determine the limit function f by evaluating
$$f(z) = \lim_{n \to \infty} f_n(z), \text{ for } z \in E;$$
(b) find a null sequence $\{a_n\}$ of positive terms such that
$$|f_n(z) - f(z)| \le a_n,$$
for $n = 1, 2, \ldots$, and all $z \in E$.

4. If $\{\phi_n\}$ is a sequence of functions, then the series of functions
$$\sum_{n=1}^{\infty} \phi_n = \phi_1 + \phi_2 + \cdots$$
converges pointwise/uniformly on a set E if the sequence of **partial sum functions**
$$f_n(z) = \phi_1(z) + \phi_2(z) + \cdots + \phi_n(z)$$
converges pointwise/uniformly on E, respectively. The limit function of the sequence $\{f_n\}$ is called the **sum function** of the series $\sum_{n=1}^{\infty} \phi_n$ on E, written
$$f(z) = \sum_{n=1}^{\infty} \phi_n(z) \quad (z \in E).$$

5. **Weierstrass' M-test**
Let $\{\phi_n\}$ be a sequence of functions defined on a set E and suppose that there is a sequence of positive terms $\{M_n\}$, such that
1. $|\phi_n(z)| \le M_n$, for $n = 1, 2, \ldots$, and all $z \in E$;
2. $\sum_{n=1}^{\infty} M_n$ is convergent.

Then the series $\sum_{n=1}^{\infty} \phi_n$ is uniformly convergent on E.

6. **Weierstrass' Theorem**
Let $\{f_n\}$ be a sequence of functions which are analytic on a region $\mathcal{R}$ and which converge uniformly to a function f on each closed disc in $\mathcal{R}$. Then
$$f \text{ is analytic on } \mathcal{R}$$
and
$$\text{the sequence } \{f_n'\} \text{ converges uniformly to } f'$$
on each closed disc in $\mathcal{R}$.

7. When Weierstrass' Theorem is used to prove that a function f defined by a series is analytic, the derivative f' may be obtained by term by term differentiation of the series.

8. The **zeta function** ζ, defined by
$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \quad (\text{Re } z > 1),$$
is analytic, and can be continued analytically to $\mathbb{C} - \{1\}$.

Section 4: The Gamma Function

1. Let $\mathcal{R}$ be a region and let K be a complex-valued function of the two variables $z \in \mathcal{R}$ and $t \in [a, b]$, such that

  1. K is analytic on $\mathcal{R}$ as a function of z, for each $t \in [a, b]$;

  2. K and $\partial K/\partial z$ are continuous on $[a, b]$ as functions of t, for each $z \in \mathcal{R}$;

  3. for some $M > 0$,

$$|K(z, t)| \le M, \quad \text{for } z \in \mathcal{R}, t \in [a, b].$$

Then the function

$$f(z) = \int_a^b K(z, t)\, dt \quad (z \in \mathcal{R})$$

is analytic on $\mathcal{R}$ and

$$f'(z) = \int_a^b \frac{\partial K}{\partial z}(z, t)\, dt, \quad \text{for } z \in \mathcal{R}.$$

2. The gamma function Γ, defined by

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt,$$

is such that

  (a) Γ is analytic on $H = \{z : \operatorname{Re} z > 1\}$,

  (b) $\Gamma(n) = (n-1)!$, for $n > 1$,

  (c) $\Gamma(z+1) = z\Gamma(z)$, for $z \in H$.

The identity in part (c) is called the **functional equation of the gamma function**.

3. The gamma function has an analytic continuation Γ to $\mathbb{C} - \{0, -1, -2, \ldots\}$ with simple poles at $0, -1, -2, \ldots$, such that

$$\operatorname{Res}(\Gamma, -k) = \frac{(-1)^k}{k!}, \quad \text{for } k = 0, 1, 2, \ldots.$$

The functional equation of the gamma function holds on $\mathbb{C} - \{0, -1, -2, \ldots\}$.

4. The value of the gamma function at $\frac{5}{2}$ is

$$\Gamma\left(\tfrac{5}{2}\right) = \tfrac{1}{2}\sqrt{\pi}.$$

5. An alternative definition of the gamma function is

$$\Gamma(z) = \lim_{n \to \infty} \frac{n! \, n^z}{z(z+1)\cdots(z+n)},$$
$$\text{for } z \in \mathbb{C} - \{0, -1, -2, \ldots\}.$$
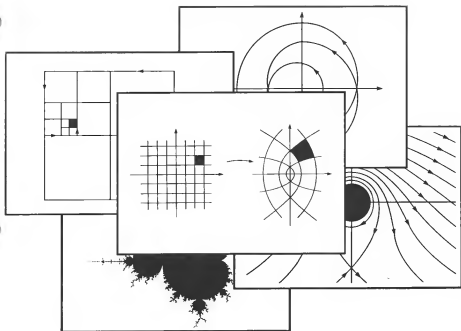
Section 5: Riemann's Legacy

This is a reading-only section.

PART II: INDEX

The Open University

COMPLEX ANALYSIS

HANDBOOK FOUR

CONTENTS

PART I: UNIT SUMMARIES
Unit D1   Conformal Mappings

Section 1: Linear and Reciprocal Functions

1. A function of the form $f(z) = az + b$, where $a, b \in \mathbb{C}$ and $a \neq 0$, is called a linear function.

2. Linear functions map circles onto circles and straight lines onto straight lines. Furthermore,

   (a) given any two circles C_1 and C_2, there is a linear function that maps C_1 onto C_2;

   (b) given any two lines L_1 and L_2, there is a linear function that maps L_1 onto L_2.

3. The function $f(z) = 1/z$ is called the reciprocal function.

4. Strategy for finding the equation of the image of a path under $f(z) = 1/z$

   To find the equation of the image $f(\Gamma)$ of a path Γ under $f(z) = 1/z$:

   (a) write down an equation that relates the x- and y-coordinates of all points $z = x + iy$ on Γ;

   (b) replace x by $\dfrac{u}{u^2 + v^2}$ and y by $\dfrac{-v}{u^2 + v^2}$;

   (c) simplify the resulting equation to obtain an equation that relates the u- and v-coordinates of all points $u + iv$ on the image $f(\Gamma)$.

5. Every line or circle has an equation of the form

   $$a(x^2 + y^2) + bx + cy + d = 0,$$

   where $a, b, c, d \in \mathbb{R}$ and $b^2 + c^2 > 4ad$.
   Conversely, any such equation represents a line or a circle.
   Also

   (a) the equation represents a line if and only if $a = 0$;

   (b) the line or circle passes through the origin if and only if $d = 0$.

6. The extended complex plane $\widehat{\mathbb{C}}$ is the union of the ordinary complex plane $\mathbb{C}$ and one extra point, the point at infinity, which is denoted by ∞; thus $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

7. A function f has a removable singularity, a pole of order k, or an essential singularity at ∞, if the function $g(u) = f(1/u)$ has the corresponding type of singularity at 0.

8. Let f be a rational function. The extended function $\widehat{f}$ associated with f is the extension of f to $\widehat{\mathbb{C}}$ obtained by defining, for each singularity α of f in $\widehat{\mathbb{C}}$,

   $$\widehat{f}(\alpha) = \beta,$$

   where β in $\widehat{\mathbb{C}}$ satisfies
   $$f(z) \to \beta \text{ as } z \to \alpha.$$

9. If L is a line in $\mathbb{C}$, then $L \cup \{\infty\}$ is an extended line in $\widehat{\mathbb{C}}$.

10. A generalized circle is either a circle or an extended line. It is completely determined by three distinct points on it.

11. Extended linear functions and the extended reciprocal function map

    (a) $\widehat{\mathbb{C}}$ one-one onto $\widehat{\mathbb{C}}$;

    (b) generalized circles onto generalized circles.

12. The sphere $\mathbb{S}$ of unit radius and centre the origin is called the Riemann sphere.

13. The function $\pi : \mathbb{S} \longrightarrow \widehat{\mathbb{C}}$, defined by

    $$\pi(u, v, s)$$
    $$= \begin{cases} \left(\dfrac{u}{1-s}\right) + \left(\dfrac{v}{1-s}\right)i, & (u, v, s) \in \mathbb{S} - \{(0,0,1)\}, \\ \infty, & (u, v, s) = (0,0,1), \end{cases}$$

    is called stereographic projection.

14. (a) Under stereographic projection, all ordinary circles on the Riemann sphere $\mathbb{S}$ map to generalized circles in $\widehat{\mathbb{C}}$.

    (b) Stereographic projection preserves angles.

Section 2: Möbius Transformations

1. A function of the form

   $$f(z) = \dfrac{az + b}{cz + d}, \quad \text{where } a, b, c, d \in \mathbb{C} \text{ and } ad - bc \neq 0,$$

   is called a Möbius transformation.

2. Every Möbius transformation is analytic and conformal.

3. The extended function associated with a Möbius transformation is called the extended Möbius transformation.

4. Every Möbius transformation is either a linear function, or a composition of linear functions and the reciprocal function.

5. Extended Möbius transformations map

   (a) $\widehat{\mathbb{C}}$ one-one onto $\widehat{\mathbb{C}}$;

   (b) generalized circles onto generalized circles.

3

6. **Inverse function of a Möbius transformation**

The Möbius transformation

$$f(z) = \frac{az+b}{cz+d}, \quad \text{where } a, b, c, d \in \mathbb{C} \text{ and } ad - bc \neq 0,$$

is a one-one function from $\mathbb{C} - \{-d/c\}$ onto $\mathbb{C} - \{a/c\}$, with inverse function

$$f^{-1}(w) = \frac{dw - b}{-cw + a}.$$

Furthermore, $\widehat{f}^{-1} = \widehat{f^{-1}}$.

7. The set of extended Möbius transformations has the following group properties.

Closure If $\widehat{f}$ and $\widehat{g}$ are extended Möbius transformations, then so is $\widehat{f} \circ \widehat{g}$.

Identity The identity function on $\widehat{\mathbb{C}}$ is an extended Möbius transformation.

Inverses Every extended Möbius transformation $\widehat{f}$ has an inverse function $\widehat{f}^{-1}$.

Associativity If $\widehat{f}, \widehat{g}$ and $\widehat{h}$ are extended Möbius transformations, then

$$\widehat{f} \circ (\widehat{g} \circ \widehat{h}) = (\widehat{f} \circ \widehat{g}) \circ \widehat{h}.$$

8. A point α in the extended complex plane $\widehat{\mathbb{C}}$ is a fixed point of an extended Möbius transformation $\widehat{f}$ if $\widehat{f}(\alpha) = \alpha$.

9. An extended Möbius transformation, other than the identity function, has either one or two fixed points in $\widehat{\mathbb{C}}$.

10. If $\widehat{f}$ and $\widehat{g}$ are two extended Möbius transformations that agree on a set consisting of three or more points in $\widehat{\mathbb{C}}$, then $\widehat{f} = \widehat{g}$.

11. Given three distinct points α, β, γ in $\widehat{\mathbb{C}}$, and any other three distinct points α', β', γ' in $\widehat{\mathbb{C}}$, there is a unique extended Möbius transformation $\widehat{f}$ that maps α to α', β to β' and γ to γ'.

In particular, if α', β', γ' is the **standard triple** of points 0, 1, ∞, respectively, then $\widehat{f}$ corresponds to

$$f(z) = \frac{(z - \alpha)}{(z - \gamma)} \frac{(\beta - \gamma)}{(\beta - \alpha)}.$$

In general, f is given by the **Implicit Formula**

$$\frac{(z - \alpha)}{(z - \gamma)} \frac{(\beta - \gamma)}{(\beta - \alpha)} = \frac{(w - \alpha')}{(w - \gamma')} \frac{(\beta' - \gamma')}{(\beta' - \alpha')}.$$

12. If C_1 and C_2 are generalised circles, then there is an extended Möbius transformation that maps C_1 onto C_2.

Section 3: Images of Generalized Circles

1. Although the images under an extended Möbius transformation of three distinct points on a generalised circle C do determine the image of C, it may be difficult to identify that image.

2. The **Apollonian form** of the equation of a generalised circle is

$$|z - \alpha| = k|z - \beta|, \quad \text{where } \alpha, \beta \in \mathbb{C} \text{ and } k > 0.$$

If $k = 1$, the generalised circle is an extended line.

3. Let C be a generalised circle. Then α and β are inverse points with respect to C if

EITHER α and β are equal and lie on C,

OR there exists an extended Möbius transformation $\widehat{f}$ that maps α to 0, β to ∞, and C to the unit circle.

4. The points α and β in $\widehat{\mathbb{C}}$ are inverse points with respect to a generalised circle C if and only if

EITHER both α and $\beta \in \mathbb{C}$ and C has the equation $|z - \alpha| = k|z - \beta|$, for some $k > 0$,

OR one of the points (β say) is ∞ and C has the equation

$$|z - \alpha| = r, \quad \text{for some } r > 0.$$

5. If α is the centre of a circle C, then α and ∞ are inverse points with respect to C.

6. Let $\widehat{f}$ be an extended Möbius transformation. If α and β are inverse points with respect to the generalised circle C, then $\widehat{f}(\alpha)$ and $\widehat{f}(\beta)$ are inverse points with respect to $\widehat{f}(C)$.

7. Let C be the generalised circle with equation $|z - \alpha| = k|z - \beta|$, where $\alpha, \beta \in \mathbb{C}$ and $k > 0$.

   (a) If $k \neq 1$, then C is a circle with centre λ and radius r, where

   $$\lambda = \frac{\alpha - k^2\beta}{1 - k^2} \quad \text{and} \quad r = \frac{k|\alpha - \beta|}{|1 - k^2|}.$$

   Also, λ lies on the line through α and β, and

   $$(\alpha - \lambda)\overline{(\beta - \lambda)} = r^2. \qquad (*)$$

   (b) If $k = 1$, then C is the line through $\frac{1}{2}(\alpha + \beta)$ perpendicular to $\beta - \alpha$.

8. The process of applying the function $f(z) = 1/\overline{z}$ is known as **inversion**.

9. Let C be a generalised circle and let β be an arbitrary point of $\widehat{\mathbb{C}}$. Then β has a unique inverse point α with respect to C, which can be found by applying Equation $(*)$ if C is a circle or by reflection if C is an extended line.
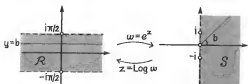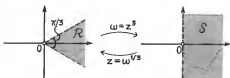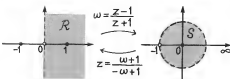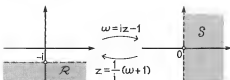
Section 4: Transforming Regions

1. An open disc centred at ∞ is a set of the form
$$\{z : |z| > M\} \cup \{\infty\}.$$

2. Let A be a subset of $\hat{\mathbb{C}}$ and let $\alpha \in \hat{\mathbb{C}}$. Then α is a boundary point of A in $\hat{\mathbb{C}}$ if each open disc centred at α contains at least one point of A and at least one point of $\hat{\mathbb{C}} - A$.

   The set of boundary points in $\hat{\mathbb{C}}$ of A forms the boundary in $\hat{\mathbb{C}}$ of A.

3. Let f be a Möbius transformation and let $\mathcal{R}$ be a region in the domain of f. Then $f(\mathcal{R})$ is a region and the extended Möbius transformation $\hat{f}$ maps the boundary in $\hat{\mathbb{C}}$ of $\mathcal{R}$ onto the boundary in $\hat{\mathbb{C}}$ of $f(\mathcal{R})$.

4. Any generalized circle divides $\hat{\mathbb{C}}$ into two parts, each of which is called a generalized open disc.

5. Conformal mappings between basic regions may be used to find, by composition, conformal mappings between other regions.

   Examples of one-one conformal mappings from $\mathcal{R}$ onto $\mathcal{S}$ and their inverse functions are given in the following figures.

6. The restriction of the function tan to
$$\mathcal{R} = \{z : -\pi/2 < \mathrm{Re}\, z < \pi/2\}$$
has inverse function $\tan^{-1}$ defined by
$$\tan^{-1} w = \frac{1}{2i} \mathrm{Log}\left(\frac{1+iw}{1-iw}\right) \quad (w \in \mathcal{S}),$$
where $\mathcal{S} = \mathbb{C} - \{iv : |v| \geq 1\}$.

7. The restriction of the function sin to
$$\mathcal{R} = \{z : -\pi/2 < \mathrm{Re}\, z < \pi/2\}$$
has inverse function $\sin^{-1}$ defined by
$$\sin^{-1} w = \frac{1}{i} \mathrm{Log}(iw + \sqrt{1 - w^2}) \quad (w \in \mathcal{S}),$$
where $\mathcal{S} = \mathbb{C} - \{u \in \mathbb{R} : |u| \geq 1\}$.

8. The Joukowski function J is defined by
$$J(z) = z + \frac{1}{z} \quad (z \neq 0).$$
The restriction of J to $\{z : |z| > 1\}$ has inverse function J^{-1} defined by
$$J^{-1}(w) = \frac{1}{2}\left(w + w\sqrt{1 - 4/w^2}\right) \quad (w \in \mathbb{C} - [-2, 2]).$$

Section 5: The Riemann Mapping Theorem

This is a reading-only section.

$$\omega = iz - 1$$
$$z = \frac{1}{i}(\omega + 1)$$

$$\omega = \frac{z - 1}{z + 1}$$
$$z = -\frac{\omega + 1}{-\omega + 1}$$

$$\omega = z^3$$
$$z = \sqrt[3]{z}$$

$$\omega = e^z$$
$$z = \mathrm{Log}\, \omega$$

Unit D2   Fluid Flows

Section 1: Setting up the Model

1. Basic Mathematical Model

We assume that

(i) the fluid forms a continuum, and any spatial variation of the flow velocity is continuous;

(ii) the flow is two-dimensional;

(iii) the flow is steady.

With these assumptions, we may represent the **flow velocity** at all points by a continuous complex function q, whose domain is the region occupied by the fluid.

2. If $q(z_0) = 0$, then z_0 is a **stagnation point** of the flow with velocity function q.

3. If q is a constant function, then the associated flow is a **uniform flow**.

4. A streamline through the point z_0, for a flow with velocity function q, is a smooth path $\Gamma : \gamma(t)$ $(t \in I)$ such that

(a) $\gamma'(t) = q(\gamma(t))$, for $t \in I$;

(b) $z_0 = \gamma(t_0)$, for some $t_0 \in I$.

If $q(z_0) = 0$, then $\{z_0\}$ is a **degenerate streamline**, with constant parametrization

$$\gamma(t) = z_0 \quad (t \in I).$$

5. The component of $q(z)$ in the direction specified by $e^{i\theta}$ is

$$q_\theta(z) = \operatorname{Re}\left(\overline{q(z)}e^{i\theta}\right).$$

The component of $q(z)$ in the direction specified by $e^{i(\theta - \pi/2)}$ is

$$q_{(\theta - \pi/2)}(z) = \operatorname{Im}\left(\overline{q(z)}e^{i\theta}\right).$$

6. The **conjugate velocity function** $\overline{q}$ is given by

$$\overline{q}(z) = \overline{q(z)}.$$

7. If $\Gamma : \gamma_1(t)$ $(t \in [a, b])$ is a smooth path of length L, then $\gamma_1(s)$ $(s \in [0, L])$ is an equivalent parametrization. A parametrization such as this, with

$$|\gamma_1'(s)| = 1, \quad \text{for } 0 \le s \le L,$$

is called a **unit-speed parametrization**.

8. If $\Gamma : \gamma(s)$ $(s \in [0, L])$ is a smooth path with unit-speed parametrization, which lies in the region $\mathcal{R}$ of a flow velocity function q, then, for each $s \in [0, L]$, the flow velocity $q(\gamma(s))$ has

(a) tangential component $q_T(s)$ in the direction specified by $\gamma'(s)$;

(b) normal component $q_N(s)$ in the direction specified by $-i\gamma'(s)$.

9. The **circulation** of q along Γ is

$$C_\Gamma = \int_0^L q_T(s)\, ds,$$

and the **flux** of q across Γ is

$$F_\Gamma = \int_0^L q_N(s)\, ds.$$

10. $$C_\Gamma + iF_\Gamma = \int_\Gamma \overline{q}(z)\, dz.$$

11. A velocity function with domain a region $\mathcal{R}$ is

(a) **locally circulation-free** if $C_\Gamma = 0$ for each simple-closed contour Γ in $\mathcal{R}$ whose inside lies in $\mathcal{R}$;

(b) **locally flux-free** if $F_\Gamma = 0$ for each simple-closed contour in $\mathcal{R}$ whose inside also lies in $\mathcal{R}$.

12. A model flow of a fluid is described by a continuous complex velocity function (whose domain is a region) which is locally circulation-free and locally flux-free.

13. A fluid flow is a **model flow** on a region $\mathcal{R}$ if and only if

$$\int_\Gamma \overline{q}(z)\, dz = 0,$$

for each simple-closed contour Γ in $\mathcal{R}$ whose inside also lies in $\mathcal{R}$.

14. A steady two-dimensional fluid flow with continuous velocity function q on a region $\mathcal{R}$ is a model flow if and only if the conjugate velocity function $\overline{q}$ is analytic on $\mathcal{R}$.

15. Let q be a model flow velocity function with domain a region $\mathcal{R}$, and let D be a punctured open disc in $\mathcal{R}$ with centre α. Then

(a) α is a **source** of strength $\mathcal{F}$ if $\mathcal{F}_\Gamma = \mathcal{F} = \mathcal{C}$ for each simple-closed contour Γ in D which surrounds α;

(b) α is a **vortex** of strength $\mathcal{C}$ if $C_\Gamma = \mathcal{C} = \mathcal{C}$ for each simple-closed contour Γ in D which surrounds α.

A source of negative strength is a **sink**.

16. Let q be a continuous velocity function on a region $\mathcal{R}$, and suppose that $q_1 = \operatorname{Re} q$ and $q_2 = \operatorname{Im} q$ have partial derivatives with respect to x and y which are continuous. Then q is

(a) **locally circulation-free** if and only if

$$\frac{\partial q_2}{\partial x} - \frac{\partial q_1}{\partial y} = 0 \text{ on } \mathcal{R};$$

(b) **locally flux-free** if and only if

$$\frac{\partial q_1}{\partial x} + \frac{\partial q_2}{\partial y} = 0 \text{ on } \mathcal{R}.$$

Section 2: Complex Potential Functions

1. Let q be a model flow velocity function with domain $\mathcal{R}$. A function Ω which is a primitive of $\overline{q}$ is called a **complex potential function** for the flow.

   Moreover,
   $$\mathcal{C}_\Gamma + i\mathcal{F}_\Gamma = \int_\Gamma \Omega'(z)\,dz = \Omega(\beta) - \Omega(\alpha),$$

   where Γ is any contour lying in the domain of Ω with initial and final points α and β.

   Such a complex potential function Ω always exists on a simply-connected subregion of $\mathcal{R}$, by the Primitive Theorem.

2. If Ω is a complex potential function on a simply-connected subregion S of the flow region $\mathcal{R}$, then
   $$\operatorname{Im}(\Omega(z)) \text{ is constant along each streamline within } S.$$

3. Each point of the flow region has just one streamline through it.

4. If $\Omega(z) = \Phi(x, y) + i\Psi(x, y)$, with $\Psi = \operatorname{Im}\Omega$ the **stream function** for the model flow with complex potential function Ω and the family of curves given by
   $$\Psi(x, y) = \text{constant}$$

   forms the streamlines of the flow.

5. There is no flux across a streamline, and so a streamline of given flow can act as a boundary of another flow.

Section 3: Flow past an Obstacle

1. Some standard notation:
   $$K_a = \{z : |z| \le a\};$$
   $$\Omega_{a,c}(z) = z + \frac{a^2}{z} - ic \operatorname{Log} z;$$
   $$q_{a,c}(z) = \overline{\Omega'_{a,c}(z)} = 1 - \frac{a^2}{z^2} - \frac{ic}{z};$$
   where $a, c \in \mathbb{R}$, with $a > 0$.

2. For $a > 0$, $c \in \mathbb{R}$, the model flow velocity function $q_{a,c}$ satisfies the following properties:
   (a) $\lim\limits_{z \to \infty} q_{a,c}(z) = 1$;
   (b) ∂K_a is the union of streamlines for $q_{a,c}$;
   (c) for any simple-closed contour surrounding K_a,
   (i) $\mathcal{C}_\Gamma = \operatorname{Re} \int_\Gamma \overline{q_{a,c}(z)}\,dz = 2\pi c$,
   and
   (ii) $\mathcal{F}_\Gamma = \operatorname{Im} \int_\Gamma \overline{q_{a,c}(z)}\,dz = 0$.

3. An **obstacle** is a compact connected set K in $\mathbb{C}$, such that $\mathbb{C} - K$ is also connected.

4. **Obstacle Problem**

   Given an obstacle K and a real number c, we seek a model flow velocity function q with domain the region $\mathcal{R} = \mathbb{C} - K$, satisfying the following properties:

   (a) $\lim\limits_{z \to \infty} q(z) = 1$;

   (b) there is a complex potential function Ω for q on either $\mathcal{R}$ or $\mathcal{R} - \Sigma$, where Σ is a simple smooth path in $\mathcal{R}$, and a real constant k such that for each $\alpha \in \partial K$, we have $\lim\limits_{z \to \alpha} \operatorname{Im}(\Omega(z)) = k$;

   (c) for any simple-closed contour Γ in $\mathcal{R}$ surrounding K,
   $$\mathcal{C}_\Gamma = 2\pi c.$$

5. For the flow in the Obstacle Problem, $2\pi c$ is called the **circulation** of the flow.

6. The model flow velocity function $q_{a,c}$ solves the Obstacle Problem for $K = K_a$, with circulation $2\pi c$ around K.

7. **Flow Mapping Theorem**

   Let K be an obstacle and let f be a one-one conformal mapping from $\mathbb{C} - K$ onto $\mathbb{C} - K_a$, where $a > 0$, such that
   $$f(z) = z + a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots, \quad \text{for } |z| > R,$$
   where $R > 0$ and $a_0, a_{-1}, a_{-2}, \ldots \in \mathbb{C}$. Then the velocity function
   $$q(z) = q_{a,c}(f(z))\overline{f'(z)} \quad (z \in \mathbb{C} - K)$$
   is the unique solution to the Obstacle Problem for K, with complex potential function
   $$\Omega = \Omega_{a,c} \circ f.$$

8. The function
   $$J_a(z) = z + \frac{a^2}{z} \quad (z \in \mathbb{C} - \{0\}),$$
   where $a > 0$, has the inverse function
   $$J_a^{-1}(w) = \tfrac{1}{2}\left(w + w\sqrt{1 - 4a^2/w^2}\right)$$
   $$(w \in \mathbb{C} - [-2a, 2a]).$$

9. A point at which a function f has zero derivative is a **critical point** of f.

10. A **Joukowski aerofoil** (in the z-plane) is an obstacle which (possibly after an appropriate translation or rotation) has boundary $J_a(B)$, where B is a closed disk passes through the critical point $w = a$ of the function $J_a(w) = w + a^2/w$ and surrounds the other critical point $w = -a$.

    The point $z = 2a$ is called the **trailing edge** of the aerofoil.

11. A symmetric aerofoil in a uniform stream in the direction of the positive x-axis has angle of attack ϕ, if ϕ is the angle measured clockwise from the negative x-axis to the line of symmetry of the aerofoil.

12. The restriction of the function

$$J_\alpha(z) = z + \frac{\alpha^2}{z} \quad (z \in \mathbb{C} - \{0\}),$$

where $\alpha \in \mathbb{C}$, $\alpha \neq 0$, to $\mathbb{C} - K_{[\alpha]}$ is one-one and has inverse function

$$J_\alpha^{-1}(w) = \tfrac{1}{2}\left(w + w\sqrt{1 - 4\alpha^2/w^2} \right)$$

$$(w \in \mathbb{C} - L(-2\alpha, 2\alpha)),$$

where $L(-2\alpha, 2\alpha)$ is the line segment between -2α and 2α.

13. The following identity is often useful:

$$J_\alpha^{-1}(w) + \alpha^2/J_\alpha^{-1}(w) = w.$$

Section 4: Lift and Drag

1. In a fluid of constant density ρ, the relationship between fluid pressure $p(z)$ and fluid speed $|q(z)|$ is given by Bernoulli's Equation

$$p(z) + \tfrac{1}{2}\rho|q(z)|^2 = p_0,$$

where the constant p_0 is the fluid pressure at any stagnation point.

2. The total force F acting on an obstacle K whose boundary is a simple-closed smooth path of length L is

$$F = i \int_0^L p(\gamma(s)) \overline{\gamma'(s)} \, ds,$$

where $\gamma : [0, L] \longrightarrow \partial K$ is a unit-speed parametrization of ∂K.

3. If F is the total force acting on an obstacle, due to a fluid whose velocity function satisfies the Obstacle Problem, then

(a) $\operatorname{Re} F$ is called the drag on the obstacle;

(b) $\operatorname{Im} F$ is called the lift on the obstacle.

4. Blasius' Theorem

Let q be a solution to the Obstacle Problem for an obstacle K, whose boundary ∂K is a simple-closed smooth path, and suppose that q has a continuous extension to ∂K. Then the conjugate of the total force acting on K is

$$\overline{F} = \tfrac{1}{2}i\rho \int_{\partial K} (\overline{q}(z))^2 \, dz,$$

where ρ is the density of the fluid.

5. If K is an obstacle whose boundary is not a simple-closed smooth path, then the conjugate of the total force acting on K is

$$\overline{F} = \tfrac{1}{2}i\rho \int_{\Gamma} (\overline{q}(z))^2 \, dz,$$

where Γ is any simple-closed contour surrounding K.

6. Kutta–Joukowski Lift Theorem

Let q be a solution to the Obstacle Problem for an obstacle K, with circulation $2\pi c$ around K. Then the obstacle experiences a total force

$$F = -2\pi c\rho i.$$

7. Kutta–Joukowski Hypothesis

The circulation around a Joukowski aerofoil is such that the flow velocity q is bounded throughout the flow region, and $q(z)$ tends to a limit as z approaches the trailing edge of the aerofoil.

Unit D3   The Mandelbrot Set

Section 1: Iteration of Analytic Functions

1. A sequence $\{z_n\}$ defined by
$$z_{n+1} = f(z_n), \quad n = 0, 1, 2, \ldots,$$
where f is a function, is called an **iteration sequence** with initial term z_0.

2. The nth **iterate** of a function f is the function obtained by applying the function f exactly n times:
$$f^n = f \circ f \circ \cdots \circ f.$$
Also, f^0 denotes the identity function $f^0(z) = z$.

3. A point α is a **fixed point** of f if
$$f(\alpha) = \alpha.$$
The equation $f(z) = z$ is called the **fixed point equation**.

4. Let α be a fixed point of an analytic function f and suppose that $|f'(\alpha)| < 1$. Then there exists $r > 0$ such that
$$\lim_{n \to \infty} f^n(z_0) = \alpha, \quad \text{for } |z_0 - \alpha| < r.$$

5. The fixed point α of an analytic function f is
   (a) **attracting**, if $|f'(\alpha)| < 1$;
   (b) **repelling**, if $|f'(\alpha)| > 1$;
   (c) **indifferent**, if $|f'(\alpha)| = 1$;
   (d) **super-attracting**, if $f'(\alpha) = 0$.

6. If α is an attracting fixed point of an analytic function f, then the **basin of attraction** of α under f is the set
$$\{z : f^n(z) \to \alpha \text{ as } n \to \infty\}.$$

7. Two functions f and g are **conjugate** to each other if
$$g = h \circ f \circ h^{-1},$$
for some one-one function h called the **conjugating function**. If the sequence $\{z_n\}$ is defined by
$$z_{n+1} = f(z_n), \quad n = 0, 1, 2, \ldots,$$
for some z_0, and $w_n = h(z_n)$, for $n = 0, 1, 2, \ldots$, then the sequence $\{w_n\}$ satisfies
$$w_{n+1} = g(w_n), \quad \text{for } n = 0, 1, 2, \ldots,$$
and $\{z_n\}$ and $\{w_n\}$ are called **conjugate iteration sequences**.

8. Let p be a polynomial function. Then $\{z_n\}$,
$$z_{n+1} = z_n - \frac{p(z_n)}{p'(z_n)}, \quad n = 0, 1, 2, \ldots,$$
is the **Newton–Raphson iteration sequence**, and the function N, where
$$N(z) = z - \frac{p(z)}{p'(z)},$$
is the **Newton–Raphson function** corresponding to p.

Section 2: Iterating Complex Quadratics

1. The iteration sequence
$$z_{n+1} = a z_n^2 + b z_n + c, \quad n = 0, 1, 2, \ldots,$$
where $a \neq 0$, is conjugate to the iteration sequence
$$w_{n+1} = w_n^2 + d, \quad n = 0, 1, 2, \ldots,$$
where $d = ac + \frac{1}{2}b - \frac{1}{4}b^2$. The conjugating function is
$$h(z) = az + \frac{1}{2}b.$$

2. The set of functions $\{P_c : c \in \mathbb{C}\}$ defined by
$$P_c(z) = z^2 + c,$$
where $c \in \mathbb{C}$, is the **family of basic quadratic functions**.

3. Let $r_c = \frac{1}{2} + \sqrt{\frac{1}{4} + |c|}$. Then, for $|z_0| > r_c$,
$$\{|P_c^n(z_0)|\} \text{ is an increasing sequence, and}$$
$$P_c^n(z_0) \to \infty \text{ as } n \to \infty.$$

4. For $c \in \mathbb{C}$, the **escape set** E_c is
$$E_c = \{z : P_c^n(z) \to \infty \text{ as } n \to \infty\}.$$
The **keep set** K_c is the complement of E_c.

5. A set A is **completely invariant** under a function f if
$$z \in A \iff f(z) \in A.$$

6. For each $c \in \mathbb{C}$, the escape set E_c and the keep set K_c have the following properties:
   (a) $E_c \supseteq \{z : |z| > r_c\}$ and $K_c \subseteq \{z : |z| \leq r_c\}$;
   (b) E_c is open and K_c is closed;
   (c) $0 \in E_c$ and $K_c \neq \varnothing$;
   (d) E_c and K_c are each completely invariant under P_c;
   (e) E_c and K_c are each symmetric under rotation by π about 0;
   (f) E_c is connected and K_c has no holes in it.

7. The point α is a **periodic point**, with period p, of a function f if
$$f^p(\alpha) = \alpha, \quad \text{and } f^k(\alpha) \neq \alpha, \text{ for } k = 1, 2, \ldots, p-1.$$
The p points
$$\alpha, f(\alpha), f^2(\alpha), \ldots, f^{p-1}(\alpha)$$
then form a **cycle** of period p, or a **p-cycle** of f. All the periodic points of P_c lie in the keep set K_c.

9

8. Let α, $f(\alpha)$, $f^2(\alpha)$, ..., $f^{p-1}(\alpha)$ form a p-cycle of an analytic function f. Then

   (a) $(f^p)'(\alpha) = f'(\alpha) \times f'(f(\alpha)) \times f'(f^2(\alpha))$
$$\times \cdots \times f'(f^{p-1}(\alpha))$$

and, moreover,

   (b) the derivative of f^p takes the same value at each point of the p-cycle; that is,

$$(f^p)'(\alpha) = (f^p)'(f(\alpha)) = (f^p)'(f^2(\alpha))$$
$$= \ldots = (f^p)'(f^{p-1}(\alpha)).$$

9. Let α belong to a p-cycle of an analytic function f. The number $(f^p)'(\alpha)$ is the *multiplier* of the p-cycle.

10. If α is a periodic point, with period p, of an analytic function f, then α and the corresponding p-cycle is

   (a) attracting, if $|(f^p)'(\alpha)| < 1$;

   (b) repelling, if $|(f^p)'(\alpha)| > 1$;

   (c) indifferent, if $|(f^p)'(\alpha)| = 1$;

   (d) super-attracting, if $(f^p)'(\alpha) = 0$.

11. Let α be a periodic point of the function P_c.

   (a) If α is attracting, then α is an interior point of K_c.

   (b) If α is repelling, then α is a boundary point of K_c.

12. The Julia set J_c of P_c is the boundary of K_c.

K_c is called the 'filled-in Julia set'.

Section 3: Graphical Iteration

1. Graphical iteration with a real function f is the process of constructing the sequence $\{x_n\}$ where
$$x_{n+1} = f(x_n), \quad n = 0, 1, 2, \ldots,$$
for a given x_0, by drawing alternately vertical and horizontal lines joining the points
$$(x_0, 0) \text{ to } (x_0, x_1) \quad (= (x_0, f(x_0)))$$
$$(x_0, x_1) \text{ to } (x_1, x_1)$$
$$(x_1, x_1) \text{ to } (x_1, x_2) \quad (= (x_1, f(x_1)))$$
$$(x_1, x_2) \text{ to } (x_2, x_2)$$
$$(x_2, x_2) \text{ to } (x_2, x_3) \quad (= (x_2, f(x_2)))$$
and so on,
using the graphs $y = f(x)$ and $y = x$.

2. If $c \in \mathbb{R}$, then the real function P_c has

   (a) no real fixed points if $c > \frac{1}{4}$;

   (b) the single fixed point $\frac{1}{2}$ if $c = \frac{1}{4}$;

   (c) the two real fixed points $\frac{1}{2} \pm \sqrt{\frac{1}{4} - c}$, if $c < \frac{1}{4}$.

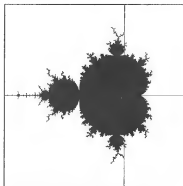**3.** If $c > \frac{1}{4}$, then $K_c \cap \mathbb{R} = \varnothing$.

4. If $-2 \leq c \leq \frac{1}{4}$, then $K_c \cap \mathbb{R} = I_c$, where
$$I_c = \left[-\frac{1}{2} - \sqrt{\frac{1}{4} - c}, \ \frac{1}{2} + \sqrt{\frac{1}{4} - c} \right].$$

5. If $c < -2$, then the set $K_c \cap \mathbb{R}$ consists of the closed interval I_c from which a sequence of disjoint, non-empty, open subintervals of I_c has been removed. In particular, $0 \notin K_c$.

Section 4: The Mandelbrot Set

1. A set A is *disconnected* if there are disjoint open sets G_1 and G_2 such that
$$A \cap G_1 \neq \varnothing, \quad A \cap G_2 \neq \varnothing \quad \text{and} \quad A \subseteq G_1 \cup G_2.$$
A set A is *connected* if it is not disconnected.

2. Any pathwise connected set is connected.

3. The *Mandelbrot set* is the set M of complex numbers c such that K_c is connected.

The square represented in this figure is
$$\{c : 0.5 \leq \text{Re } c \leq 1, -1.5 \leq \text{Im } c \leq 1.5\}.$$

4. For any $c \in \mathbb{C}$,
$$K_c \text{ is connected} \iff 0 \in K_c.$$

5. The Mandelbrot set M can be specified as follows:
$$M = \{c : |P_c^n(0)| \leq 2, \text{ for } n = 1, 2, \ldots \}.$$

6. The Mandelbrot set M

   (a) is a compact subset of $\{c : |c| \leq 2\}$;

   (b) is symmetric under reflection in the real axis;

   (c) meets the real axis in the interval $[-2, \frac{1}{4}]$;

   (d) has no holes in it.

7. The Mandelbrot set is connected.

8. If the function P_c has an attracting cycle, then $c \in M$.

9. (a) The function P_c has an attracting fixed point if and only if c satisfies
$$\left(8|c|^2 - \tfrac{3}{2}\right)^2 + 8\,\mathrm{Re}\,c < 3.$$

(b) The function P_c has an attracting 2-cycle if and only if c satisfies
$$|c + 1| < \tfrac{1}{4}.$$

10. A periodic region is a maximal region $\mathcal{R}$ such that, for some positive integer p,

the function P_c has an attracting p-cycle,

for all $c \in \mathcal{R}$.

11. The function P_c has a super-attracting p-cycle if and only if
$$P_c^p(0) = 0, \text{ but } P_c^k(0) \neq 0, \text{ for } k = 1, 2, \dots, p-1.$$

12. The number λ is a primitive nth root of unity if λ is a root of unity and if n is the smallest positive integer for which $\lambda^n = 1$.

13. Suppose that the function P_{c_0}, $c_0 \in \mathbb{C}$, has a p-cycle whose multiplier λ is a root of unity.

(a) Saddle-node bifurcation at c_0. If $\lambda = 1$, then c_0 is the cusp of a cardioid-shaped periodic region $\mathcal{R}$, such that

P_c has an attracting p-cycle, for $c \in \mathcal{R}$.

(b) Period-multiplying bifurcation at c_0. If λ is a primitive nth root of unity, for $n > 1$, then there are two periodic regions $\mathcal{R}_1$ and $\mathcal{R}_2$ whose boundaries meet at c_0 such that

P_c has attracting $\begin{cases} p\text{-cycle}, & \text{for } c \in \mathcal{R}_1, \\ np\text{-cycle}, & \text{for } c \in \mathcal{R}_2. \end{cases}$

14. The set of points which are mapped to a set E by P_c is called the preimage set of E and is denoted by $P_c^{-1}(E)$:
$$P_c^{-1}(E) = \{z : P_c(z) \in E\}.$$

15. A compact disc is a compact set whose boundary is a simple-closed smooth path.

16. If E is a compact disc and $c \notin \partial E$, then $P_c^{-1}(E)$ has

(a) one compact disc containing 0, if $c \in \mathrm{int}\,E$;

(b) two compact discs, neither containing 0, if $c \in \mathrm{ext}\,E$.

Section 5: Beyond the Mandelbrot Set

This is a reading-only section.

PART II: INDEX